Math 253 Exam 1 Review

Name: ____________________

NOTE: You will be allowed your calculator for the exam. However, you will also be expected to show your work/reasoning in order to earn ANY credit at all. Meaning, answers alone will earn zero points.

1. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

a. $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, ...\right\}$

b. $\{5, 1, 5, 1, 5, 1, ...\}$

2. Determine whether the following sequences are convergent or divergent. If they are convergent, find the limit.

a. $a_n = \frac{n^3}{n+1}$

c. $\{\arctan(2n)\}$

b. $a_n = \frac{(-1)^n n^3}{n^3 + 2n^2 + 1}$

d. $a_n = \frac{\sin(2n)}{1+\sqrt{n}}$

3. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + ...$$

4. Write the number $2.3\overline{17}$ as a ratio of integers.

5. Determine whether the following series converge or diverge and state which test was used and how you might know which test to use.

a. $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$

c. $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

b. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

d. $\sum_{n=1}^{\infty} \frac{4 + 3^n}{2^n}$

e. $\sum_{n=0}^{\infty} \frac{1+\sin(n)}{10^n}$

g. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$

f. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$

h. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

i. $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$

j. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. Let $R_n = s - s_n$ (that is, the remainder after the n^{th} partial sum). Then

$$\int_{n+1}^{\infty} f(x)\ dx \leq R_n \leq \int_{n}^{\infty} f(x)\ dx.$$

6. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms. Determine bounds for this estimation.

7. How many terms are necessary to ensure that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is accurate to within 0.0005?

Alternating Series Estimation Theorem: If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series with b_n decreasing and limiting out to zero, then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

8. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

9. Is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ absolutely convergent?

10. Determine the radius and interval of convergent for the following series.

    $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

11. Express the following functions as a power series and determine the interval of convergence for each.

a. $f(x) = \dfrac{x}{2x^2 + 1}$

b. $f(x) = \ln(x^2 + 4)$

12. Use a power series to approximate the definite integral to six decimal places.

$\int_0^{0.4} \ln(1 + x^4)\, dx$

13. Show that $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a solution to the differential equation $f'(x) = f(x)$.

14. Find the Taylor series for $f(x)$ centered at the given value of a. Assume that f has a power series expansion. No need to show that $R_n \to 0$.

a. $f(x) = \frac{1}{x}$, $a = -3$

b. $f(x) = \cos(3x)$, $a = 0$

15. Prove that the series obtained above for $f(x) = \cos(3x)$ not just converges, but indeed converges to $\cos(3x)$.

16. What are the 7 power series (listed in 8.7) you should be handily familiar with and of these, which ones should you have memorized?

17. Use a Maclaurin series (above) to obtain the needed Maclaurin series given.

a. $x^2 \ln(1 + x^3)$

b. $f(x) = \dfrac{x^2}{\sqrt{2 + x}}$

18. Use series to approximate $\int_0^{0.5} x^2 e^{-x^2}\, dx$ with $|\text{error}| < 0.001$.

19. Find the sum of the following series by realizing that it is a power series representation of a function being evaluated for a specific x value.

$\sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

20. For the fullowing functions, approximate f by a Taylor polynomial with degree n at the number a. Then use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval. Check your results (just on your calculator) by graphing $|R_n(x)| = |f(x) - T_n(x)|$.

a. $f(x) = x^{-2}$, $a = 1$, $n = 2$, $0.9 \leq x \leq 1.1$

b. $f(x) = \ln(1 + 2x)$, $a = 1$, $n = 3$, $0.5 \leq x \leq 1.5$

21. The resistivity ρ of a conducting wire is the reciprocal of the conductivity and is measured in units of ohm-meter (Ω-m). The resistivity of a given metal depends on the temperature according to the equation

$$\rho(t) = \rho_{20} e^{\alpha(t-20)}$$

where t is the temperature in °C. There are tables that list the values of α (called the temperature coefficient) and ρ_{20} (the resistivity at 20°C) for various metals. Except at very low temperatures, the resistivity varies almost linearly with temperature and so it is common to approximate the expression for $\rho(t)$ by its first- or second-degree Taylor polynomial at $t = 20$.

Find expression for these linear and quadratic approximations. For copper, the tables give $\alpha = 0.0039/°\text{C}$ and $\rho_{20} = 1.7 \times 10^{-8}\Omega$-m. Graph the resistivity of copper and the linear and quadratic approximations for $-250°\text{C} \leq t \leq 1000°\text{C}$. For what values of t does the linear approximation agree with the exponential expression to within one percent?