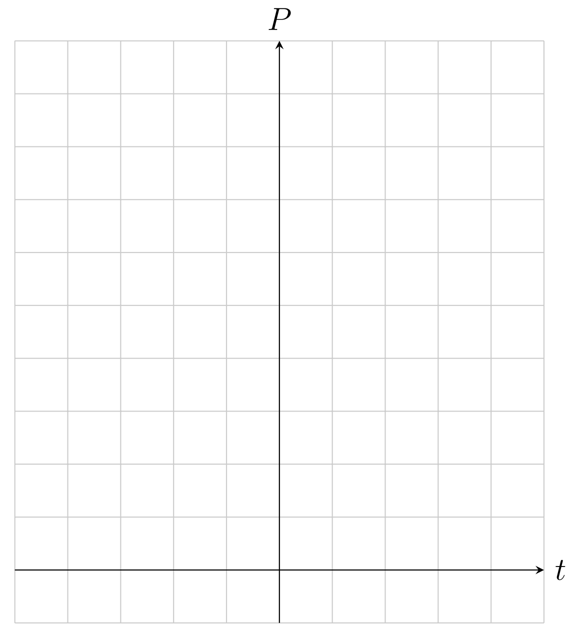
MTH 256 Final Exam Review Name:______________________________

1. Verify by substituion that every member of the family of functions $y(t) = \dfrac{1 + Ae^{t^2}}{1 - Ae^{t^2}}$ is a solution to the differential equation $y'(t) = t(y^2 - 1)$.

2. Suppose a cup of coffee is at 150°F in a 68°F room. Suppose further that, when the coffee is 120°F, the coffee is cooling at a rate of 2°F per minute. Set up a differential equation modeling this scenario, solve the differential equation in order to determine a function modeling the temperature of the coffee t-minutes from when it was 150°F. How long will it take for the coffee to reach 90°F?

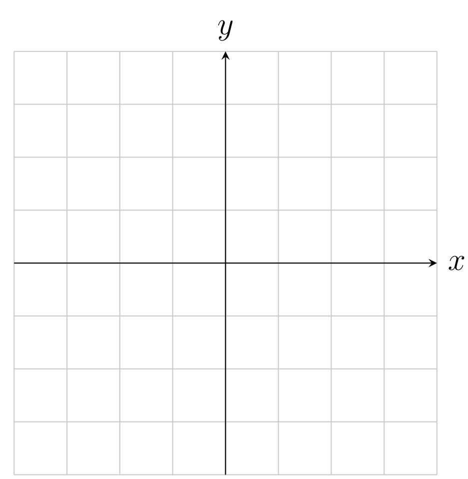
3. Given a population model of $\frac{dP}{dt} = kP\left(1 - \frac{P}{30}\right)\left(\frac{P}{50} - 1\right)\left(\frac{P}{100} - 1\right)$, draw a phase line next to a graph of specific solutions to the differential equation. What do the numbers 30 and 50 and 100 potentially represent?



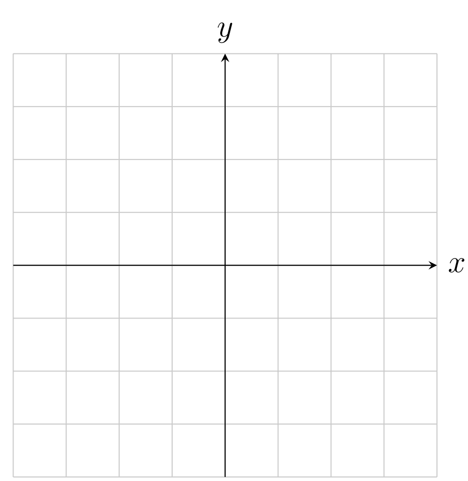
4. Given the differential equation $\frac{dy}{dt} = y(-y^2 + 9y + \alpha)$, draw a bifurcation diagram with several phase lines and cooresponding graphs of specific solutions.

5. Suppose a 200-gallon tank initially contains 100 gallons of sugar water at a concentration of 0.1 lbs of sugar per gallon. Suppose that sugar-water, with a concentration of 0.15 lbs of sugar per gallon, is poured into the tank at a rate of 5 gallons per minute, and that sugar water is removed after mixing at a rate of 3 gallons per minute. Set up and solve a differential equation modeling the amount of sugar in the tank after t-minutes. What will the concentration of sugar be when the tank is full?

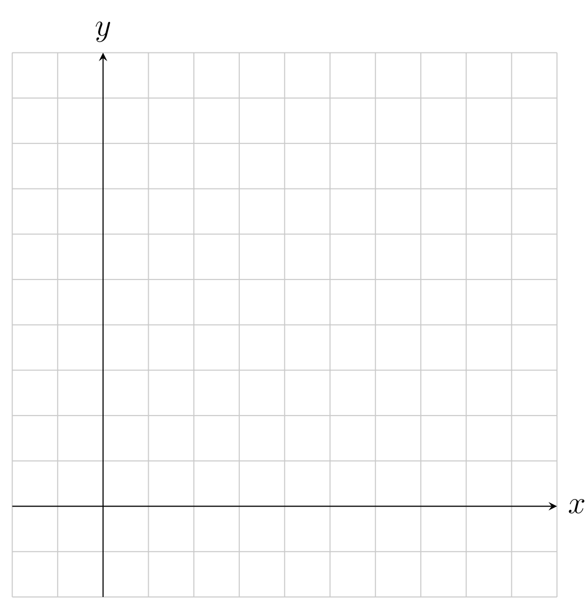
6. Given the system of differential equations $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3 & -13 \\ 4 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, compute the eigenvalues and eigenvectors then use the real and imaginary parts to put together the general solution to the system of equations. Sketch the vector field and phase portrait. Determine the particular solution with initial condition $(x(0), y(0)) = (52, 12)$.



7. Find a general solution to $\frac{d\mathbf{y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix} \mathbf{y}$. Sketch the vector field and phase portrait. Determine the particular solution with initial condition $(x(0), y(0)) = (4, -2)$



8. Given the system of differential equations $x' = 60x - 4x^2 - 3xy$, $y' = 42y - 2y^2 - 3xy$, which models the rates of changes of two interacting species populations, describe the type of x- and y-populations involved (exponential or logistic) and the nature of their interaction (competition, cooperation, or predation). Then find and characterize the system's critical points (type and stability). Determine what nonzero x- and y-populations can coexist. Then construct a phase plane portrait that enables you to describe the long term behavior of the two populations. Use https://www.geogebra.org/m/utcMvuUy to confirm your results. [Note, the app seems to require a scroll wheel on a mouse to zoom out.]



9. Suppose you have a mass-spring harmonic oscillator where $m = 3$, $c = 18$, and $k = 15$. However, an outside force is acting on the system via the function $F(t) = 3t + e^{-t}$. Determine the particular solution satisfying $x(0) = 0$ and $x'(0) = 0$.

10. Consider an RLC circuit with $R = 200$ ohms Ω, $L = 5$ henry (H), and $C = 0.001$ farad (F). At time $t = 0$, when both $I(0)$ and $Q(0)$ are zero, the switch in the circuit is closed and an alternating current given by $E(t) = 100\sin(10t)$ is allowed into the circuit. Find the current in the cuircuit and the time lag of the steady periodic current behind the voltage. See my solution to really get this last step!

11. Given the differential equation $\frac{d^2y}{dt^2} + 8y = 3\sin(3t)$, determine the general solution, the frequency of the beats, and the frequency of rapid oscillations. What is the particular solution for when $y(0) = 0$ and $y'(0) = 0$?

12. Use Laplace Transforms to determine the function modeling the current in an RLC circuit with $L = 10$ Henries, $R = 20$ ohms, $C = 0.02$ Farads, the initial charge is $Q(0) = 0$, and the initial current is $I(0) = 1$. Then, at $t = 1$ seconds, an electromotive force given by $E(t) = 5\sin(2(t-1))$ is turned on. Use the differential equation $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = u_1(t)5\sin(2(t-1))$ to find the solution for $Q(t)$ and then take its derivative to find $I(t)$. Be careful of discontinuities when taking the derivative. Graph both Q and I using your favorite software and attach them.

13. Take the initial-value problem $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = \delta_1(t)$, $y(0) = 1$, $y'(0) = 0$. Determine the solution to this using previous methods for $0 \leq t < 1$ and graph both $y(t)$ and $v(t)$. Then use Laplace Transforms to solve the differential equation and graph both $y(t)$ and $v(t)$.