Name: Solutions

0.1 No Calculator Allowed

Exercise 1 Evaluate the following limits. Show some work or give some justification but you do NOT need to show limit laws.

2.3 ex 10 a. $\lim_{x\to\infty} \frac{\sin(x)}{x^2+x-1}$

$-1 \le \sin(x) \le 1$

$\frac{-1}{x^2+x-1} \le \frac{\sin(x)}{x^2+x-1} \le \frac{1}{x^2+x-1}$

$\lim_{x\to\infty} \frac{-1}{x^2+x-1} \le \lim_{x\to\infty} \frac{\sin(x)}{x^2+x-1} \le \lim_{x\to\infty} \frac{1}{x^2+x-1}$

$0 \le \lim_{x\to\infty} \frac{\sin(x)}{x^2+x-1} \le 0$

$\Rightarrow \lim_{x\to\infty} \frac{\sin(x)}{x^2+x-1} = 0$

ch 2.4 ex 7 d. $\lim_{x\to\pi} e^{\sin^2(x)} = e^{\sin^2(\pi)}$

$= e^0$

$= 1$

2.5 ex 5 b. $\lim_{t\to-\infty} \frac{3t^4+2t-1}{-5t^4+2t^2-1}$

$= \lim_{t\to-\infty} \frac{3t^4}{-5t^4}$

$= -\frac{3}{5}$

ch 2.5 ex 1 e. $\lim_{x\to1} \frac{x^2-x-6}{x^2-4x+3} = \lim_{x\to1} \frac{(x-3)(x+2)}{(x-3)(x-1)}$

$= \lim_{x\to1} \frac{x+2}{x-1}$ DNE

$\lim_{x\to1^-} \frac{x+2}{x-1} = -\infty$

$\lim_{x\to1^+} \frac{x+2}{x-1} = \infty$

Scratch

$\lim_{x\to1^-} \frac{x+2}{x-1} = \frac{3}{-0} = -\infty$

$\lim_{x\to1^+} \frac{x+2}{x-1} = \frac{3}{+0} = +\infty$

2.5 ex 1 c. $\lim_{x\to3} \frac{x^2+2x-3}{x^2-6x+9}$

$= \lim_{x\to3} \frac{(x+3)(x-1)}{(x-3)^2}$

$= \infty$

Scratch

$\frac{6\cdot2}{(\pm0)^2} = +\infty$

ch 3.6 ex 4, ch 2.5 ex 2 f. $\lim_{x\to(\frac{\pi}{2})^-} \ln\left(\frac{1}{\tan(x)}\right)$

$= \lim_{A\to\infty} \ln\left(\frac{1}{A}\right)$

$= \lim_{B\to0^+} \ln(B)$

$= -\infty$

Let $A = \tan(x)$

Note $A\to\infty$ as $x\to(\frac{\pi}{2})^-$

Let $B = \frac{1}{A}$

Note $B\to0^+$ as $A\to\infty$

Exercise 2 Let $f(x) = \begin{cases} \dfrac{(1-x)(x+2)}{x+2} & \text{if} \quad x \leq -1 \\ \dfrac{1}{x} & \text{if} \quad -1 < x < 1 \\ x^2 - 2x & \text{if} \quad 1 \leq x \end{cases}$

Evaluate the following and then answer the given questions.

a. i. $\lim_{x \to -3^-} f(x) = 4$ ii. $\lim_{x \to -3^+} f(x) = 4$ iii. $f(-3) = 4$

iv. Is f continuous from the left at -3?

Yes

v. Is f continuous from the right at -3?

Yes

vi. Is f continuous at -3?

Yes

vii. If f is discontinuous at -3, what kind of discontinuity is it?

N/A

b. i. $\lim_{x \to -2^-} f(x) = 3$ ii. $\lim_{x \to -2^+} f(x) = 3$ iii. $f(-2)$ is und.

iv. Is f continuous from the left at -2?

No

v. Is f continuous from the right at -2?

No

vi. Is f continuous at -2?

No

vii. If f is discontinuous at -2, what kind of discontinuity is it?

Hole (Removable Discontinuity)

c. i. $\lim_{x \to -1^-} f(x) = 2$ ii. $\lim_{x \to -1^+} f(x) = -1$ iii. $f(-1) = 2$

iv. Is f continuous from the left at -1?

Yes

v. Is f continuous from the right at -1?

No

vi. Is f continuous at -1?

No

vii. If f is discontinuous at -1, what kind of discontinuity is it?

Jump

More on next page.

d. i. $\lim_{x\to 0^-} f(x) = -\infty$ ii. $\lim_{x\to 0^+} f(x) = \infty$ iii. $f(0)$ is undefined

iv. Is f continuous from the left at 0?

No

v. Is f continuous from the right at 0?

No

vi. Is f continuous at 0?

No

vii. If f is discontinuous at 0, what kind of discontinuity is it?

Vertical Asymptote

e. i. $\lim_{x\to 1^-} f(x) = 1$ ii. $\lim_{x\to 1^+} f(x) = -1$ iii. $f(1) = -1$

iv. Is f continuous from the left at 1?

No

v. Is f continuous from the right at 1?

Yes

vi. Is f continuous at 1?

No

vii. If f is discontinuous at 1, what kind of discontinuity is it?

Jump

f. Is f continuous *only* from the right **at** $x = 1$ *or* on the interval $[1, \infty)$?

At $x=1$. If $x>1$ it is continuous from both sides

g. If I ask where f is continuous *only* from the left, is it okay to write $x \leq -1$? Why or why not?

It is NOT okay. You must write at $x=-1$ because when $x<-1$ it is continuous from both sides.

Hint for the last two problems: Parse out the definition of continuity and one sided continuity in chapter 2.4. Pay attention to the word ONLY in the above questions.

Exercise 3 Sketch the graph of a function that satisfies all of the given conditions.

i. $\lim_{x\to-\infty} f(x) = -3$, $\lim_{x\to\infty} f(x) = \infty$

ii. $f(x)$ is increasing on its entire domain $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

iii. $f''(x) < 0$ on $(-3, -2) \cup (0, 2)$

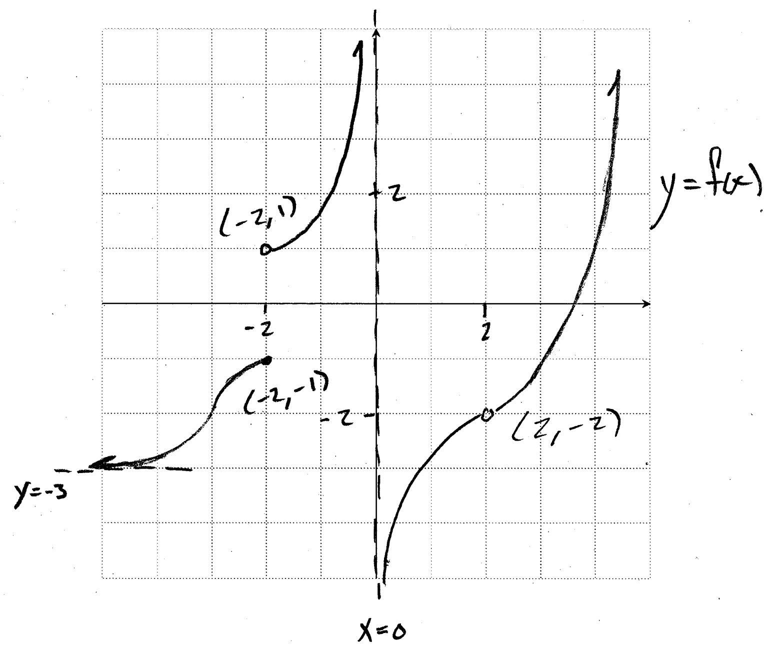
iv. $f''(x) > 0$ on $(-\infty, -3) \cup (-2, 0) \cup (2, \infty)$

v. The only discontinuities on f occur at -2, 0 and 2.

vi. There is a vertical asymptote at $x = 0$.

vii. There is a removable discontinuity at $x = 2$

viii. f is continuous only from the left at $x = -2$



Exercise 4 Determine the derivative function for the following. If you can simplify do so but it should be your last worry as I am much more concerned with your remembering how to take the derivatives.

a. $y = \arctan\left(x^2 + 2x\right)$

$$y' = \frac{1}{1+(x^2+2x)^2} \cdot (2x+2)$$

$$= \frac{2x+2}{1+(x^2+2x)^2}$$

c. $f(x) = x^{\left(x^3\right)}$ $= e^{x^3 \ln(x)}$

$$f'(x) = e^{x^3 \ln(x)}\left(3x^2 \ln(x) + x^3 \cdot 1/x\right)$$

$$= x^{(x^3)}\left(3x^2 \ln(x) + x^2\right)$$

$$= x^{x^3+2}\left(3\ln(x) + 1\right)$$

b. $y = (2x^3 - x) \cdot \sin(x)$

$$y' = (6x^2 - 1)\sin(x) + (2x^3 - x)\cos(x)$$

d. Find $\dfrac{dy}{dx}$ if
$x(t) = t^3$

$y(t) = \dfrac{t^{\frac{3}{2}} - 2}{3}$ $= \frac{1}{3}t^{3/2} - 2/3$
are parametric functions.

Your final answer will have t in it not x.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

$$= \frac{\frac{1}{2}t^{1/2}}{3t^2}$$

$$= \frac{\sqrt{t}}{6t^2}$$

Exercise 5 A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	3	5	7
3	7	2	7	9

a. If $h(x) = f(g(x))$, find $h'(2)$

$h'(x) = f'(g(x)) \cdot g'(x)$
$h'(2) = f'(g(2)) \cdot g'(2)$
$= f'(3) \cdot 7$
$= 7 \cdot 7 = 49$

b. If $H(x) = g(f(x))$, find $H'(2)$

$H'(x) = g'(f(x)) \cdot f'(x)$
$H'(2) = g'(f(2)) \cdot f'(2)$
$= g'(1) \cdot 5$
$= 6 \cdot 5 = 30$

Exercise 6 Find an equation of the tangent line to $f(x) = 3x^2 - 4x + 1$ when $x = 1$.

$f'(x) = 6x - 4$

$f(1) = 3 - 4 + 1 = 0$

$f'(1) = 2$

$T_{f(1)}(x) = 2(x-1) + 0$
$= 2x - 2$

Exercise 7 Let $f(x) = 2xe^x$.

a. Determine $f''(x)$

$f'(x) = 2e^x + 2xe^x$

$f''(x) = 2e^x + 2e^x + 2xe^x$
$= 4e^x + 2xe^x$

b. Determine the interval for which $f''(x) < 0$.

$4e^x + 2xe^x < 0$
$2e^x(2 + x) < 0$
$2 + x < 0$
$x < -2$

$(-\infty, -2)$

c. On what interval is $f(x)$ concave down?

On $(-\infty, -2)$

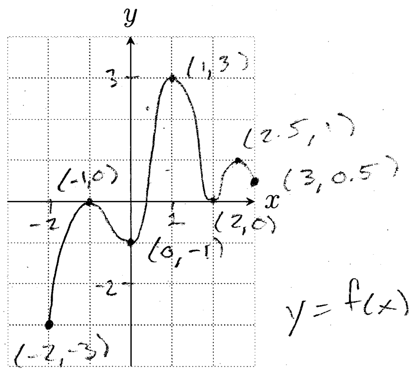
Exercise 8 Sketch the graph of a function f that is continuous on $[-2, 3]$ and has the given properties:

i. Local minimums at 0 and 2.

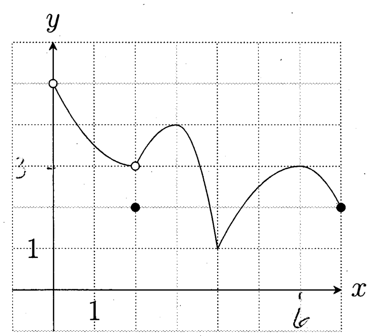
ii. Absolute minimum at -2.

iii. Local maximums at -1 and 2.5. & 1

iv. Absolute maximum at 1.



Exercise 9 For the given graph, state where any local and absolute minimums and maximums occur and what those minimum and maximum values are.



local mins: 2 @ 2
1 @ 4

local maxs: 4 @ 3
3 @ 6

Abs min: 1 @ 4

Abs max: None

Exercise 10 Find the critical numbers for the following functions.

a. $f(x) = x^{\frac{3}{4}} - 2x^{\frac{1}{4}}$ $\quad D_f = [0, \infty)$

$f'(x) = \frac{3}{4}x^{-1/4} - \frac{1}{2}x^{-3/4}$ $\quad D_{f'} = (0, \infty)$

$= \frac{1}{4}x^{-3/4}(3x^{1/2} - 2)$

$3x^{1/2} - 2 = 0$

$x^{1/2} = 2/3$

$x = 4/9$

Crits: $0, 4/9$

b. $g(t) = \dfrac{t-1}{t^2+4}$ $\quad D_g = (-\infty, \infty)$

$g'(t) = \dfrac{t^2+4 - (t-1)\cdot 2t}{(t^2+4)^2}$ $\quad D_{g'} = (-\infty, \infty)$

$= \dfrac{-t^2+2t+4}{(t^2+4)^2}$

$0 = t^2 - 2t - 4$

$t = \dfrac{+2 \pm \sqrt{4+16}}{2} = \dfrac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$

Crits: $1 \pm \sqrt{5}$

Exercise 11 Find the absolute maximum and absolute minimum values of $f(x) = x - \ln(x)$ on the closed interval $\left[\frac{1}{2}, 2\right]$.

$f'(x) = 1 - \frac{1}{x}$

$0 = 1 - \frac{1}{x}$

$\frac{1}{x} = 1$

$1 = x$

$f(\frac{1}{2}) = \frac{1}{2} - \ln(\frac{1}{2}) \approx 1.19$

$f(1) = 1$

$f(2) = 2 - \ln(2) \approx 1.31$

Abs min $= 1$ @ $x = 1$

Abs max $= 2 - \ln(2)$ @ $x = 2$

I'll have either a problem where no calculator is needed or put this in the calc okay section of the final

Exercise 12 For the function $f(x) = 3x^{\frac{2}{3}} - x$, determine the intercepts, find any vertical or horizontal asymptotes, the intervals of increase and decrease, the local maximum and minimum values and where the occur, the intervals of concavity and inflection points, the long run behavior of the graph, organize the information along a number line as shown in class or in a table as shown in the text and lab-manual, and then graph the function.

$f(x) = 3\sqrt[3]{x^2} - x$ $\quad D_f = (-\infty, \infty)$

$f'(x) = 2x^{-1/3} - 1$

$= \frac{2}{\sqrt[3]{x}} - 1$ $\quad D_{f'} = (-\infty, 0) \cup (0, \infty)$

$0 = \frac{2}{\sqrt[3]{x}} - 1$ $\quad$ Crits: $0, 8$

$1 = \frac{2}{\sqrt[3]{x}}$

$\sqrt[3]{x} = 2$

$x = 8$

		0		8	
f	⌒		⌒		⌒
f'	$-$		$+$		$-$
f''	$-$		$-$		$-$

$f''(x) = -\frac{2}{3}x^{-4/3}$

$= -\frac{2}{3\sqrt[3]{x^4}}$ $\quad D_{f''} = (-\infty, 0) \cup (0, \infty)$

$0 = -\frac{2}{3\sqrt[3]{x^4}}$ No sols.

No V.A.s

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 3x^{2/3} - x = \lim_{x \to \infty} x^{2/3}(3 - x^{1/3})$

$= -\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^{2/3}(3 - x^{1/3})$

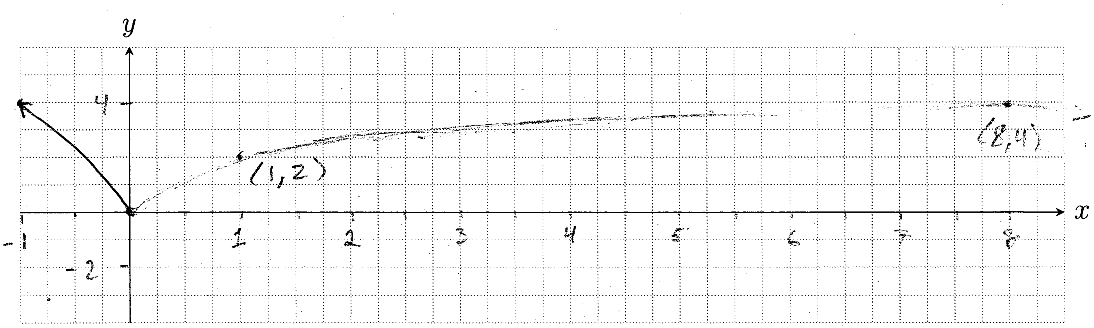
$= -\infty$

$f(0) = 0$

$f(8) = 4$

$f(1) = 2$

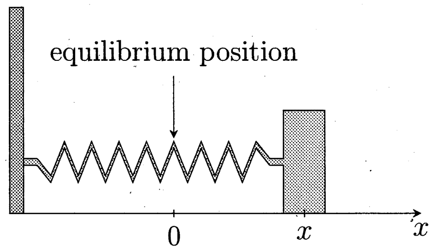
$f(-1) = 4$



$y = f(x)$

0.2 Calculator Allowed

Exercise 13 A mass on a spring vibrates horizontally on a smooth level surface (see figure). Its equation of motion is $x(t) = -2\cos(t)$, where t is in seconds and x in centimeters.



a. Find the velocity and acceleration functions at time t.

$v(t) = 2\sin(t)$

$a(t) = 2\cos(t)$

At $t = \frac{2\pi}{3}$ s the mass is 1cm to the right of equilibrium, moving to the right at about 1.73 cm/s & accelerating to the left 1 cm/s².

b. Determine the times at which the velocity will be $-1\frac{cm}{s}$.

$-1 = 2\sin(t)$

$\sin(t) = -\frac{1}{2}$

$t = \frac{7\pi}{6} + 2\pi k, \quad t = \frac{11\pi}{6} + 2\pi k$

The velocity will be −1 cm/s at $\frac{7\pi}{6}$ s & $\frac{11\pi}{6}$ s & every multiple of 2π s before and after these times.

c. Find the position, velocity, and acceleration of the mass at time $t = \frac{2\pi}{3}$. In what direction is it moving at that time?

$x(2\pi/3) = -2\cos(2\pi/3) = -2(-\frac{1}{2}) = 1$

$v(2\pi/3) = 2\sin(2\pi/3) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73$

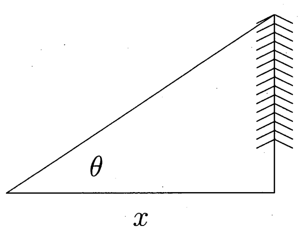
$a(2\pi/3) = 2\cos(2\pi/3) = -1$

Exercise 14 Suppose you are running towards a tree and your distance from the tree, x in meters, is a function of time, t in seconds. We may then look at the angle of inclination, θ in radians, from your eyes to the top of the tree (see picture) as a function of your distance from the tree, x still in meters. Suppose $x(3) = 24$.

If $\left.\dfrac{dx}{dt}\right|_{t=3\text{sec}} = -2\dfrac{\text{m}}{\text{sec}}$

and $\left.\dfrac{d\theta}{dx}\right|_{x=24\text{m}} \approx -0.016\dfrac{\text{rads}}{\text{m}}$,

state the value and meaning of $\left.\dfrac{d\theta}{dt}\right|_{t=3\text{sec}}$ including units.



$$\left.\frac{d\theta}{dt}\right|_{t=3s} = \left.\frac{d\theta}{dx}\right|_{t=3} \cdot \left.\frac{dx}{dt}\right|_{t=3}$$

$$= \left.\frac{d\theta}{dx}\right|_{x=24} \cdot (-2)$$

since $x(3)=24$ ⟶

$$\approx -0.016(-2) = 0.032 \text{ rads/s}$$

Three seconds into the run the angle of inclination as you look at the top of the tree is increasing at about 0.032 rad/s.

Exercise 15 The displacement (in meters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 4\sin(\pi(t - .5))$, where t is measured in seconds.

a. Find the average velocity during each time period, organizing your work in a table. Be sure your table has proper headings that make sense in context.

$t_1 = 0$

i. $[0, 0.1]$
ii. $[0, 0.01]$
iii. $[0, 0.001]$
iv. $[0, -0.1]$
v. $[0, -0.01]$
vi. $[0, -0.001]$

t_2	$s(t_2)$	$v_{ave} = \frac{s(t_2) - s(0)}{t_2 - 0} = \frac{s(t_2)+4}{t_2}$
0.1	-3.804	1.9577
0.01	-3.998	0.1974
0.001	-3.99998	0.0197
-0.001	-3.99998	-0.0197
-0.01	-3.998	-0.1974
-0.1	-3.804	1.9577

b. Use the table to estimate the instantaneous velocity of the particle when $t = 0$.

The instantaneous velocity at $t=0$ appears to be 0 m/s.

Exercise 16 If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume of water remaining in the tank after t minutes as

$$V = 5000(1 - \tfrac{1}{40}t)^2 \text{ on } 0 \le t \le 40.$$

Find the rate at which water is draining from the tank after (a) 5 minutes, (b) 10 minutes, (c) 20 minutes, and (d) 40 minutes. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

$V'(t) = 10{,}000(1 - \frac{1}{40}t)(-\frac{1}{40})$

$= -250(1 - \frac{1}{40}t)$

$V'(5) = -250(7/8) \approx -218.75$

$V'(10) = -250(3/4) \approx -187.5$

$V'(20) = -250(1/2) = -125$

$V'(0) = -250 \quad V'(40) = 0$

The water is draining from the tank at a rate about 218.75 gal/min at 5 min, 187.5 gal/min at 10 min, & 125 gal/min at 20 min.

The tank is draining the fastest at time zero & the slowest at the end, slowing down all along the way.

Exercise 17 Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2} = GmM\,r^{-2}$$

where G is the gravitational constant and r is the distance between the bodies.

a. Find $\dfrac{dF}{dr}$ and explain its meaning. What does the minus sign indicate?

$$\frac{dF}{dr} = -\frac{2GmM}{r^3}$$

This gives the rate at which the gravitational force between 2 objects changes as their distance, r, increases. The negative indicates the force decreases as the distance increases.

b. Suppose it is known that the earth attracts an object with a force that decreases at the rate 2 N/km when $r = 20{,}000$ km. How fast does this force change when $r = 10{,}000$ km?

$-2 = \dfrac{-2GmM}{(20{,}000)^3} \Rightarrow GmM = (20{,}000)^3$

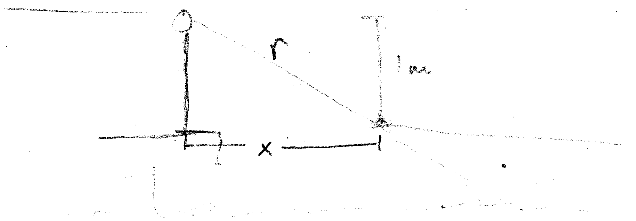
$$\frac{dF}{dr} = -\frac{2(20{,}000)^3}{r^3}$$

$$\left.\frac{dF}{dr}\right|_{r=10{,}000} = -2\left(\frac{20{,}000}{10{,}000}\right)^3 = -16$$

The force is decreasing at a rate of 16 N/km when the object is 10,000 km from Earth.

Exercise 18 A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 meter higher than the bow of the boat. If the rope is pulled at a rate of 1 meter per second, how fast is the boat approaching the dock when it is 8 meters from the dock?

$\leftarrow 1^{m}/_{s}$ $\qquad \frac{dr}{dt} = -1^{m}/_{s}$



$$x^2 + 1^2 = r^2 \qquad 8^2 + 1^2 = \left(r\big|_{x=8}\right)^2$$
$$2x \cdot \frac{dx}{dt} = 2r\frac{dr}{dt} \qquad \sqrt{65} = r\big|_{x=8}$$
$$\frac{dx}{dt} = -\frac{r}{x}$$

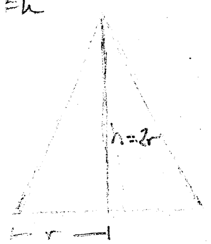
Find $\left.\frac{dx}{dt}\right|_{x=8}$

$$\left.\frac{dx}{dt}\right|_{x=8} = -\frac{\sqrt{65}}{8} \approx -1.01$$

The boat is approaching the dock at a rate of about -1.01 m/s when it's 8m from the dock.

Exercise 19 Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 feet high?

$D = h$



$r = \frac{1}{2}h$

$\frac{dV}{dt} = 30 \, ^{ft^3}/_{min}$

Find $\left.\frac{dh}{dt}\right|_{h=10}$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$
$$= \frac{\pi}{12}h^3$$
$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$
$$30 = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{120}{\pi h^2}$$
$$\left.\frac{dh}{dt}\right|_{h=10} = \frac{120}{100\pi} \approx 0.38$$

The pile's height is increasing at a rate of 0.38 ft/s when the pile is 10 ft high.