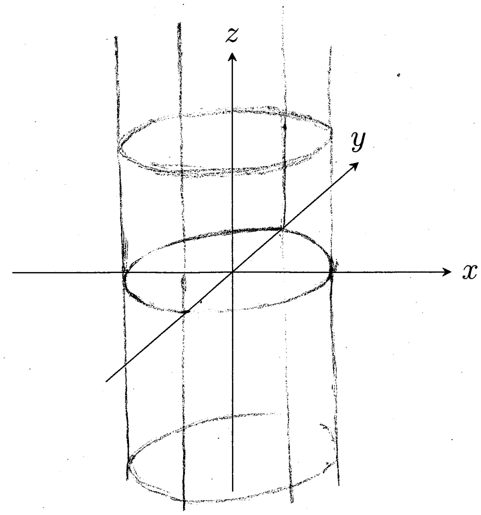
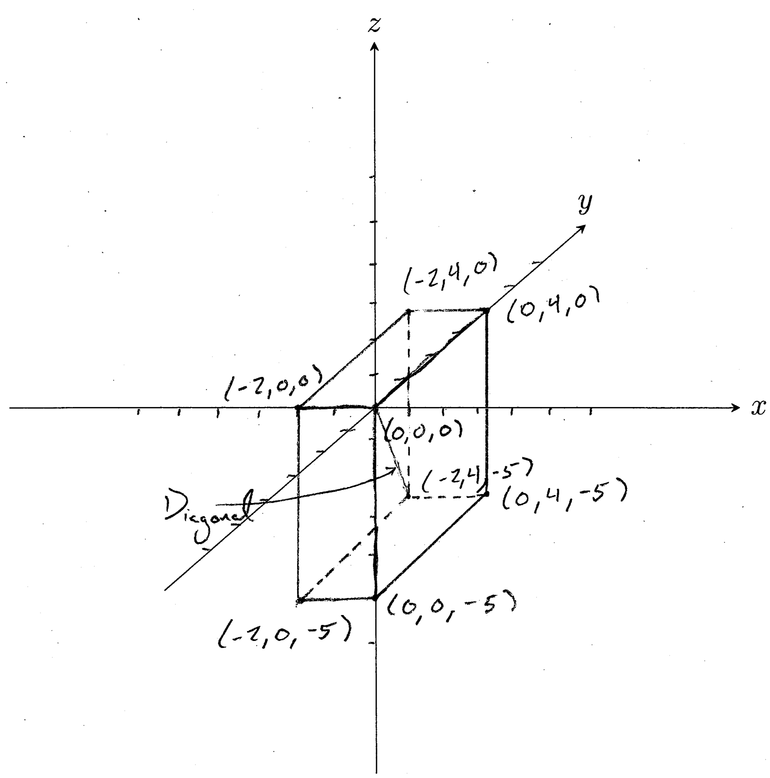
Math 253 Final Review Name: Solutions

1. Describe the set of points which satisfy the equation $x^2 + y^2 = 1$ in $\mathbb{R}^3$.



Cylinder centered around z-axis of radius 1.

2. What are the projections of the point $(-2, 4, -5)$ on the $xy-$, $yz-$ and $xz-$planes? Draw a rectangular box with the origin and $(-2, 4, -5)$ as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.



$$D = \sqrt{(-2)^2 + (4)^2 + (-5)^2}$$
$$= \sqrt{4 + 16 + 25}$$
$$= \sqrt{45}$$
$$= 3\sqrt{5}$$

3. Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

$$x^2+4x+y^2-6y+z^2+2z=-6$$

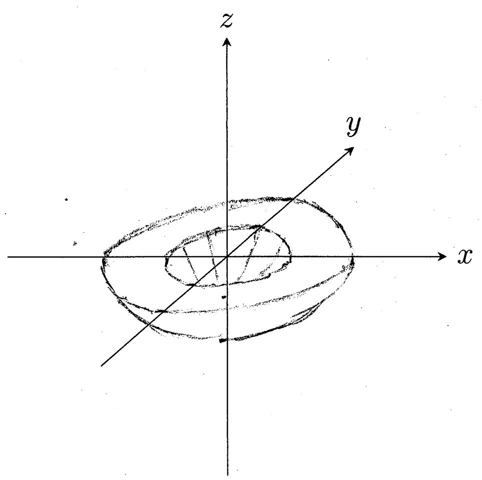
$$\quad +4 \qquad +9 \qquad +1 \qquad +14$$

$$(x+2)^2+(y-3)^2+(z+1)^2=8$$

$$C=(-2,3,-1) \qquad r=2\sqrt{2}$$

4. What region in $\mathbb{R}^3$ is represented by the inequalities

$$1 \le x^2 + y^2 + z^2 \le 4 \text{ and } z \le 0?$$



Bottom half of solid sphere with radius 2 with center sphere of radius 1 removed.

5. Find an equation of the set of all points equidistant from the points $A = (-2, 3, 3)$ and $B = (1, -3, 6)$.

Find all $P=(x,y,z)$ equidistant from A & B:

$|AP| = |BP|$

$$\sqrt{(x+2)^2+(y-3)^2+(z-3)^2} = \sqrt{(x-1)^2+(y+3)^2+(z-6)^2}$$

$$x^2+4x+4+y^2-6y+9+z^2-6z+9 = x^2-2x+1+y^2+6y+9+z^2-12z+36$$

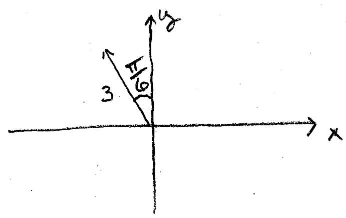
$$6x-12y+6z=24$$

$$x-2y+z=4$$

6. Write an inequality to describe the solid upper hemisphere of the sphere of radius 2 centered around the point (1,1,1).

$$(x-1)^2+(y-1)^2+(z-1)^2 \leq 4 \quad \& \quad z \geq 1$$

7. If $\mathbf{v}$ lies in the 2nd quadrant and makes an angle of $\frac{\pi}{6}$ with the positive y-axis and $|\mathbf{v}| = 3$, find $\mathbf{v}$ in component form. What angle does $\mathbf{v}$ make with the positive x-axis?



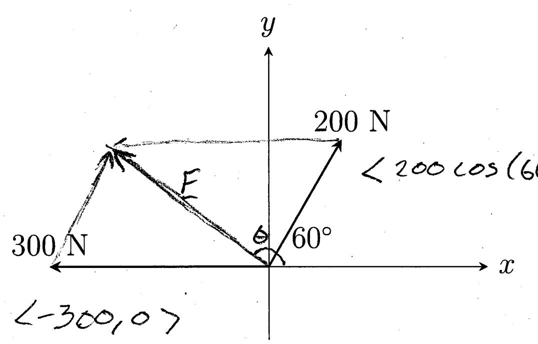
$V_x = -3\sin(\pi/6) = -3/2$

$V_y = 3\cos(\pi/6) = 3\sqrt{3}/2$

$\underline{v} = \langle -\frac{3}{2}, 3\sqrt{3}/2 \rangle$

$\underline{v}$ makes an angle of $2\pi/3$ with the x-axis.

8. Find the magnitude of the resultant force and the angle it makes with the positive x-axis.



$\langle 200\cos(60°), 200\sin(60°) \rangle = \langle 100, 100\sqrt{3} \rangle$

$\langle -300, 0 \rangle$

Resultant Force, $\underline{F} = \langle -200, 100\sqrt{3} \rangle$

$\tan(\theta) = \frac{100\sqrt{3}}{-200} = -\sqrt{3}/2$

$\theta = \tan^{-1}(-\sqrt{3}/2) + \pi \approx 2.43$ radians $\approx 139.11°$

9. Determine the angle between the following vectors.

a. $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

$$\cos(\theta) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{10-8-2}{\sqrt{9}\sqrt{45}} = 0$$

$\theta = 90°$

b. $\mathbf{a} = \langle 2, 2, -1 \rangle$, $\mathbf{b} = \langle 5, -3, 2 \rangle$.

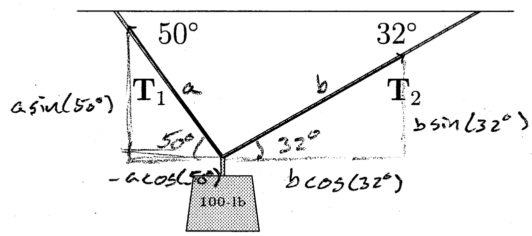
$$\cos(\theta) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{10-6-2}{\sqrt{9}\sqrt{38}} = \frac{2}{3\sqrt{38}}$$

$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 83.79°$

10. A 100-lb weight hangs from two wires as shown in the figure below. Find the tension forces $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

Let $|T_1| = a$

& $|T_2| = b$



$T_1 = \langle -a\cos(50°), a\sin(50°) \rangle$

$T_2 = \langle b\cos(32°), b\sin(32°) \rangle$

$-a\cos(50°) + b\cos(32°) = 0$

$a\sin(50°) + b\sin(32°) = 100$

$$b = \frac{a\cos(50°)}{\cos(32°)} \longrightarrow a\sin(50°) + \frac{a\cos(52°)\sin(32°)}{\cos(32°)} = 100$$

$$a\sin(50°)\cos(32°) + a\cos(52°)\sin(32°) = 100\cos(32°)$$

$$a = \frac{100\cos(32°)}{\sin(50°)\cos(32°) + \cos(52°)\sin(32°)}$$

$$= \frac{100\cos(32°)}{\sin(82°)}$$

$$\approx 85.64$$

$$b = \frac{100\cos(50°)}{\sin(82°)}$$

$$\approx 64.91$$

$T_1 \approx \langle -55.05, -65.60 \rangle$

$T_2 \approx \langle 55.05, 34.40 \rangle$

The tension in cable 1 is about 85.64 N & the tension in cable 2 is about 64.91 N.

11. A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P = (2, 1, 0)$ to the point $Q = (4, 6, 2)$. Find the work done.

$\overrightarrow{PQ} = \langle 2, 5, 2 \rangle$

$W = |\overrightarrow{PQ} \cdot F| = |\langle 2, 5, 2 \rangle \cdot \langle 3, 4, 5 \rangle|$

$= 6 + 20 + 10 = 36$

The work done is 36 units.

12. Determine the projection of **b** onto **a** for the given vectors and then determine $\text{orth}_{\mathbf{a}}\mathbf{b}$ for each.

a. $\mathbf{b} = \langle 1, 1, 2, \rangle$, $\mathbf{a} = \langle -2, 3, 1 \rangle$

$$\text{proj}_{\underline{a}}\underline{b} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\underline{a} = \frac{-2+3+2}{4+9+1}\langle -2,3,1\rangle$$

$$= \frac{3}{14}\langle -2,3,1\rangle$$

$$= \langle -3/7,\ 9/14,\ 3/14\rangle$$

$$\text{orth}_{\underline{a}}\underline{b} = \underline{a} - \text{proj}_{\underline{a}}\underline{b}$$

$$= \langle -2,3,1\rangle - \langle -3/7,\ 9/14,\ 3/14\rangle$$

$$= \langle -11/7,\ 33/14,\ 11/14\rangle$$

b. $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\text{proj}_{\underline{a}}\underline{b} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\underline{a} = \frac{1-1+1}{1+1+1}\langle 1,1,1\rangle$$

$$= \langle \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\rangle$$

$$\text{orth}_{\underline{a}}\underline{b} = \langle 1,1,1\rangle - \langle 1/3, 1/3, 1/3\rangle$$

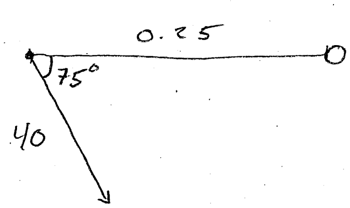
$$= \langle 2/3, 2/3, 2/3\rangle$$

13. A bolt is tightened by applying a 40-N force to a 0.25-m wrench at an angle of 75° with the arm of the wrench. Find the magnitude of the torque about the center of the bolt.

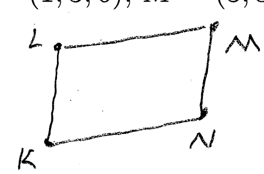
$$\tau = |40\cdot 0.25\sin(75^\circ)|$$

$$= 10\sin(75^\circ)$$

$$\approx 9.66\ \text{N}\cdot\text{m}/\text{rad}.$$



14. Find the area of the parallelogram with vertices $K = (1, 2, 3)$, $L = (1, 3, 6)$, $M = (3, 8, 6)$, and $N = (3, 7, 3)$.



$$A = |\vec{KL} \times \vec{KN}|$$

$$= |\langle 0,1,3\rangle \times \langle 2,5,0\rangle|$$

$$= \left|\begin{pmatrix}0\\1\\3\end{pmatrix} \times \begin{pmatrix}2\\5\\0\end{pmatrix}\right| = |\langle -15, 6, -2\rangle|$$

$$= \sqrt{225+36+4} = \sqrt{265} \approx 16.28$$

The area is about 16.28.

15. Find a nonzero vector orthogonal to the plane through the points $P = (-1, 3, 1)$, $Q = (0, 5, 2)$, and $R = (4, 3, -1)$ and then find the area of the triangle PQR.

$$\underline{n} = \vec{PQ} \times \vec{PR} = \begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}5\\0\\-2\end{pmatrix} = \langle -4, 7, -10\rangle$$

$$A = \tfrac{1}{2}|\langle -4, 7, -10\rangle| = \tfrac{1}{2}\sqrt{16+49+100} = \tfrac{1}{2}\sqrt{165} \approx 6.42$$

16. Determine the volume of the parallelepiped, described by the vectors $\mathbf{a} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$V = |(\underline{a} \times \underline{b}) \cdot \underline{c}| = \left| \left[\begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right] \cdot \langle -2, 2, 1 \rangle \right|$$

$$= |\langle -4, -10, 3 \rangle \cdot \langle -2, 2, 1 \rangle| = |8 - 20 + 3| = 9$$

17. Suppose that $\mathbf{a} \neq \mathbf{0}$.

a. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

No: $\langle 1,0,0 \rangle \cdot \langle 0,1,0 \rangle = 0$

& $\langle 1,0,0 \rangle \cdot \langle 0,0,1 \rangle = 0$

Yet $\langle 0,1,0 \rangle \neq \langle 0,0,1 \rangle$

b. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

No: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \langle 0,0,1 \rangle$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \langle 0,0,1 \rangle$

Yet $\langle 0,1,0 \rangle \neq \langle 2,1,0 \rangle$

c. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

Yes! $\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{c}$

$\Rightarrow \underline{a} \cdot (\underline{b} - \underline{c}) = 0$

$\Rightarrow |\underline{a}| = 0$ or $\underline{b} = \underline{c}$

or $\theta = 90°$

$\underline{a} \times \underline{b} = \underline{a} \times \underline{c} \Rightarrow \underline{a} \times (\underline{b} - \underline{c}) = \underline{0}$

$\Rightarrow \underline{a} = \underline{0}$ or $\underline{b} = \underline{c}$ or

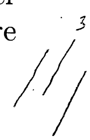
$\theta = 0°$ only overlap

since $\underline{a} \neq \underline{0}$ is $\underline{b} = \underline{c}$

18. Determine whether each statement is true or false.

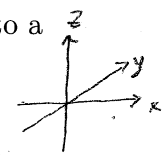
a. Two lines parallel to a third line are parallel.

True



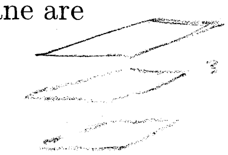
b. Two lines perpendicular to a third line are parallel.

False



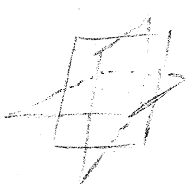
c. Two planes parallel to a third plane are parallel.

True



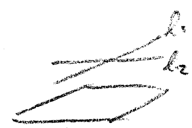
d. Two planes perpendicular to a third plane are parallel.

False



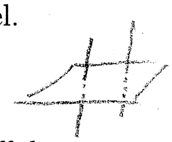
e. Two lines parallel to a plane are parallel.

False



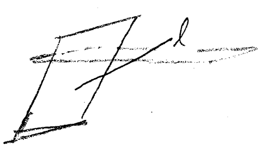
f. Two lines perpendicular to a plane are parallel.

True



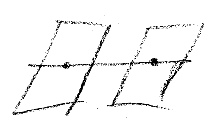
g. Two planes parallel to a line are parallel.

False



h. Two planes perpendicular to a line are parallel.

True



i. Two planes either intersect or are parallel.

True

j. Two lines either intersect or are parallel.

False: Skew Lines

k. A plane and a line either intersect or are parallel.

True

19. Find a vector equation and parametric equations for the line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$.

$$\underline{d} = \langle 2, -3, 9 \rangle$$

$$\underline{r} = \langle 0, 14, -10 \rangle + t \langle 2, -3, 9 \rangle$$

$x = 2t$
$y = 14 - 3t$
$z = -10 + 9t$

20. Find vector, parametric, and symmetric equations of the line that passes through the points $A = (2, 4, -3)$ and $B = (3, -1, 1)$. At what point does the line intersect the xy-plane?

$\vec{AB} = \langle 1, -5, 4 \rangle$ $\quad \underline{r} = \langle 2, 4, -3 \rangle + t \langle 1, -5, 4 \rangle$

$x = 2 + t \quad y = 4 - 5t \quad z = -3 + 4t$

$$x - 2 = \frac{y-4}{-5} = \frac{z+3}{4}$$

Intersects xy-plane when $z = 0$: $0 = -3 + 4t$
$\Rightarrow 3/4 = t$

$$\underline{r}(3/4) = \langle 2, 4, -3 \rangle + \tfrac{3}{4} \langle 1, -5, 4 \rangle$$
$$= \langle 2, 4, -3 \rangle + \langle 3/4, -15/4, 3 \rangle$$
$$= \langle 11/4, 1/4, 0 \rangle$$

At $P = (11/4, 1/4, 0)$

21. Find parametric equations and symmetric equations for the line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

$$\underline{d} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \langle 1, -1, 1 \rangle$$

$x = 2 + t$
$y = 1 - t$
$z = t$

$$x - 2 = \frac{y-1}{-1} = z$$

22. Determine whether the lines L_1 and L_2 described below are parallel, skew, or intersecting. If they intersect, find the point of intersection.

a.

L_1: $x = -6t$, $y = 1 + 9t$, $z = -3t$
L_2: $x = 1 + 2s$, $y = 4 - 3s$, $z = s$

$-6t = 1+2s \quad 1+9t = 4-3s \quad -3t = s$

$1+9t = 4-3(-3t)$

$-3+9t = 9t$

$-3 = 0$

No solutions

$\underline{d}_1 = \langle -6, 9, -3 \rangle$

$\underline{d}_2 = \langle 2, -3, 1 \rangle$ Parallel

b.

$$L_1: \frac{x-1}{2} = \frac{y-3}{2} = \frac{z-2}{-1}$$
$$L_2: \frac{x-2}{1} = \frac{y-6}{-1} = \frac{z+2}{3}$$

$x_1 = 1+2t \quad y_1 = 3+2t \quad z_1 = 2-t$

$x_2 = 2+s \quad y_2 = 6-s \quad z_2 = -2+3s$

$1+2t = 2+s \quad 3+2t = 6-s \quad 2-t = -2+3s$

$-1+2t = s \rightarrow 3+2t = 6-(-1+2t)$

$3+2t = 7-2t$

$4t = 4$

$t = 1$

$s = -1+2(1)$
$= 1$

$z_1 = 2-(1) = 1$ ✓
$z_2 = -2+3(1) = 1$

$x_1 = 1+2(1) = 3 = x_2$
$y_1 = 3+2(1) = 5 = y_2$

Intersect @ $(3,5,1)$

23. Find an equation of the plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$.

Point in plane: $(1,2,4)$

$\underline{n} = \langle 5, 2, 1 \rangle$

$5x + 2y + z = d$

$5(1) + 2(2) + 4 = d$

$13 = d$

$5x + 2y + z = 13$

24. a. Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

$\underline{n}_1 = \langle 1, 1, 1 \rangle \quad \underline{n}_2 = \langle 1, -2, 3 \rangle$

$$\cos(\theta) = \frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|} = \frac{1-2+3}{\sqrt{3}\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72.02^\circ$$

b. Find symmetric equations for the line of intersection L of these two planes.

$$\underline{d} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \langle 5, -2, -3 \rangle$$

$z = 0$: $x + y = 1$
$-(x - 2y = 1)$
$3y = 0$
$y = 0$

$x = 1 \quad P = (1, 0, 0)$

$\underline{r} = \langle 1, 0, 0 \rangle + t \langle 5, -2, -3 \rangle$

25. Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

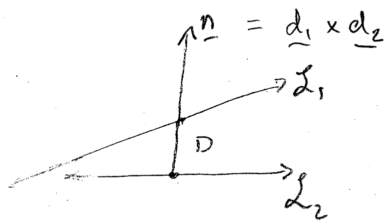
$$D = \frac{|10(0) + 2(1) - 2(0) - 5|}{\sqrt{100 + 4 + 4}}$$

Point on ↗ is $(0, 1, 0)$

$$= \frac{3}{\sqrt{108}} = \frac{3\sqrt{108}}{108} = \frac{3 \cdot 6\sqrt{3}}{108} = \frac{\sqrt{3}}{12}$$

26. Find the distance between the skew lines described by the parametric equations

L_1:	$x = 1 + t$	$y = -2 + 3t$	$z = 4 - t$
L_2:	$x = 2s$	$y = 3 + s$	$z = -3 + 4s$



$$\underline{n} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \langle 13, -6, -5 \rangle$$

So the plane containing $\mathcal{L}_2$ & parallel to $\mathcal{L}_1$

is $13x - 6y - 5z = d$ with $P_2 = (0, 3, -3)$ on $\mathcal{L}_2$ so:

$$13(0) - 6(3) - 5(-3) = d$$
$$-18 + 15 = d$$
$$-3 = d$$
$$13x - 6y - 5z = -3$$

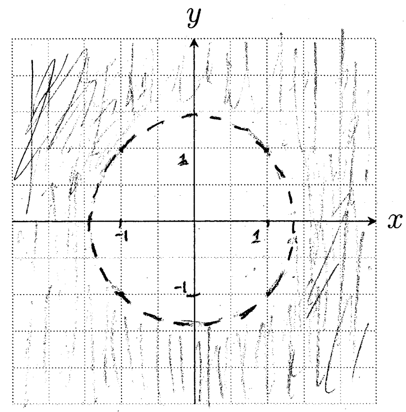
Now, since $\mathcal{L}_1$ is parallel to this plane the distance from any point on $\mathcal{L}_1$, say $(1, -2, 4)$ to this plane is the distance we're looking for:

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{169 + 36 + 25}} = \frac{8}{\sqrt{230}} \approx 0.53$$

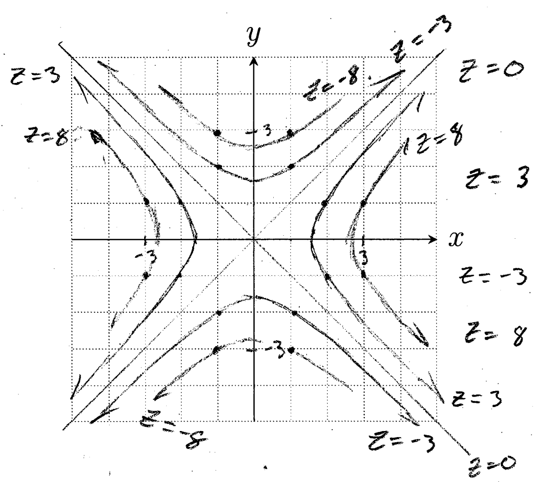
27. Find and sketch the domain of $f(x, y) = \ln(x^2 + y^2 - 2)$.

$x^2+y^2-2>0$

$D:\{(x,y) \mid x^2+y^2>2\}$



28. Use traces to sketch the graph of $f(x, y) = x^2 - y^2$



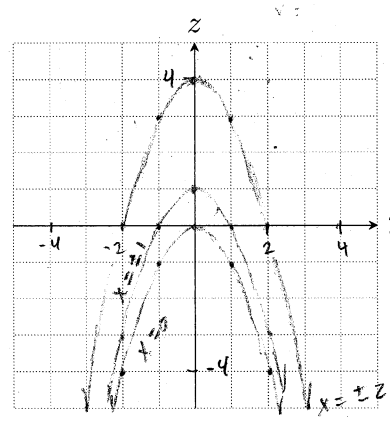
$z=0:\ x^2-y^2=0$

$y=\pm x$

$z=3:\ 3=x^2-y^2$

$z=-3:\ 3=y^2-x^2$

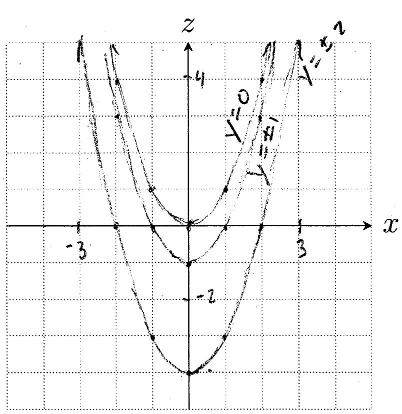
$z=8:\ 8=x^2-y^2$



$x=0:\ z=-y^2$

$x=\pm1:\ z=-y^2+1$

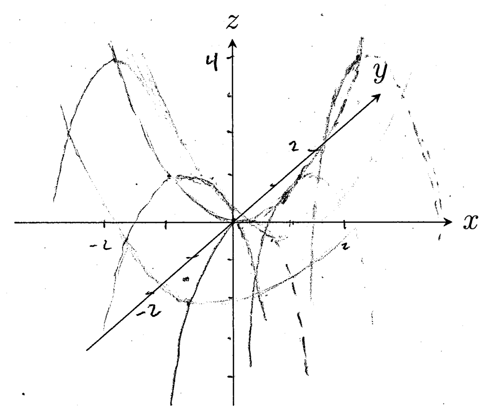
$x=\pm2:\ z=-y^2+4$



$y=0:\ z=x^2$

$y=\pm1:\ z=x^2-1$

$y=\pm2:\ z=x^2-4$



29. Classify $x^2 + 2z^2 - 6x - y + 10 = 0$ from one of the standard forms shown in chapter 9.6.

$$x^2 - 6x + 2z^2 = y - 10$$

$$(x-3)^2 \quad +9 \qquad +2z^2 = y - 1 \quad +9$$

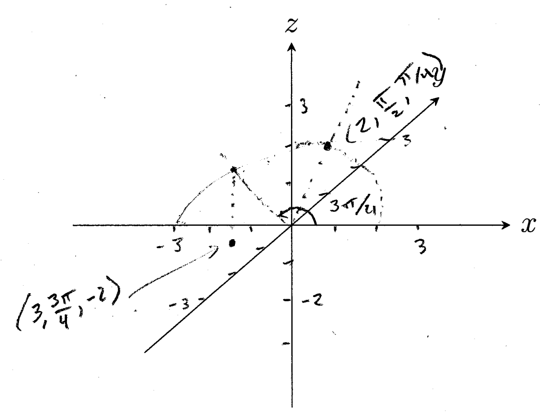
$$\frac{(x-3)^2}{2} + z^2 = \frac{y-1}{2}$$

30. Plot the points whose coordinates are given. Then find the rectangular coordinates of the point.

a. $(3, \frac{3\pi}{4}, -2)_C = \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, -2\right)$

$x = 3\cos\left(\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$

$y = 3\sin\left(\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}$



b. $(2, \frac{\pi}{2}, \frac{\pi}{4})_S = (0, \sqrt{2}, \sqrt{2})$

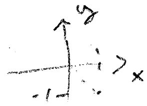
$x = 2\sin(\pi/4)\cos(\pi/2)$

$= 2 \cdot \frac{\sqrt{2}}{2} \cdot 0 = 0$

$y = 2\sin(\pi/4)\sin(\pi/2)$

$= 2 \cdot \frac{\sqrt{2}}{2} \cdot 1 = \sqrt{2}$

$z = 2\cos(\pi/4) = 2\sqrt{2}/2 = \sqrt{2}$



31. Change the point $(1, -1, 4)$ from rectangular to cylindrical coordinates and spherical coordinates.

$r = \sqrt{1+1} = \sqrt{2}$

$\tan(\theta) = \frac{-1}{1}$

$\theta = -\pi/4$

$(1,-1,4)_R = (\sqrt{2}, -\pi/4, 4)_C$

$\rho = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$

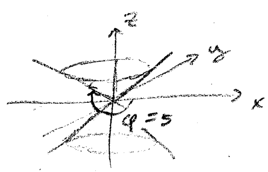
$z = \rho\cos(\varphi)$

$4 = 3\sqrt{2}\cos(\varphi)$

$\cos(\varphi) = \frac{4}{3\sqrt{2}} = \frac{4\sqrt{2}}{3\cdot 2} = \frac{2\sqrt{2}}{3}$

$\varphi = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ $\quad (1,-1,4)_R = \left(3\sqrt{2}, -\pi/4, \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right)$

32. Describe in words the surface whose equation is given by $\phi = 5$.



Cone!

33. Identify the surface whose equation is given.

a. $z = 4 - r^2$

$z = 4 - x^2 - y^2$

elliptic paraboloid downward facing with z-axis as axis of symmetry shifted up 4

b. $\rho^2(\sin^2\phi \sin^2\theta + \cos^2\phi) = 9$

$\rho^2\sin^2\varphi\sin^2\theta + \rho^2\cos^2\varphi = 9$

$y^2 + z^2 = 9$

Cylinder radius 3 with x-axis as axis of symmetry

34. Write the equation in (a) cylindrical coordinates and (b) spherical coordinates.

$x^2 - y^2 - z^2 = 1$

$r^2\cos^2\theta - r^2\sin^2\theta - z^2 = 1$

$z^2 = r^2\cos^2\theta - r^2\sin^2\theta - 1$

$z^2 = r^2(\cos^2\theta - \sin^2\theta) - 1$

$z^2 = r^2\cos(2\theta) - 1$

$\rho^2\sin^2\varphi\cos^2\theta - \rho^2\sin^2\varphi\sin^2\theta - \rho^2\cos^2\varphi = 1$

$\rho^2 = \frac{1}{\sin^2\varphi\cos^2\theta - \sin^2\varphi\sin^2\theta - \cos^2\varphi}$

35. Determine whether the following series converge or diverge and **justify** your answer.

a. $\sum_{n=1}^{\infty} \frac{\sin^2(n)+1}{4^n}$

$$0 \le \sum_{n=1}^{\infty} \frac{\sin^2(n)+1}{4^n} \le \sum_{n=1}^{\infty} \frac{2}{4^n}$$

Since $\sum_{n=1}^{\infty} \frac{2}{4^n}$ is a convergent geometric series $(r = 1/4)$, $\sum_{n=1}^{\infty} \frac{\sin^2(n)+1}{4^n}$ is convergent by direct comparison.

c. $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$

$$\lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = 1 \text{ isn't zero}$$

So $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$ diverges

By the divergence test.

b. $\sum_{n=1}^{\infty} \frac{-2}{\sqrt{1+n^4}}$

First $0 \le \sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^4}} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^4}}$ is convergent by direct comparison to a convergent p-series $(p=2)$

Thus $\sum_{n=1}^{\infty} \frac{-2}{\sqrt{1+n^4}} = -2\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^4}}$ is convergent by multiplication of real numbers.

d. $\sum_{n=1}^{\infty} \frac{(-2)^n n!}{(2n)!}$

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n \cdot n!} \right|$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{(2n+2)(2n+1)} = 0$$

Thus $\sum_{n=1}^{\infty} \frac{(-2)^n \cdot n!}{(2n)!}$ converges by the ratio test.

36. Determine the radius and interval of convergent for the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n^2+1}}$$

$(-1)^n(-1)^{n+1} = (-1)^{2n+1}$
$= -1$

$\lim_{n\to\infty} \left| \frac{x^{n+2}}{\sqrt{(n+1)^2+1}} \cdot \frac{\sqrt{n^2+1}}{x^{n+1}} \right|$

$= \lim_{n\to\infty} \left| x \cdot \sqrt{\frac{n^2+1}{n^2+2n+2}} \right|$

$= |x| < 1 \qquad R = 1$

$x = 1: \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ conv. by AST

$x = -1: \sum_{n=0}^{\infty} \frac{-1}{\sqrt{n^2+1}}$ div by comparison to $p=1$ series

$I = (-1, 1]$

37. For the following function, approximate f by a Taylor polynomial with degree n at the number a. Then use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

$$f(x) = e^x,\ a = -1,\ n = 2,\ -1.2 \le x \le -0.8$$

$d = 0.2$

$f(x) = e^x \qquad f(-1) = e^{-1}$

$f'(x) = e^x \qquad f'(-1) = e^{-1}$

$f''(x) = e^x \qquad f''(-1) = e^{-1}$

$f'''(x) = e^x$

$M = |f'''(x)|_{\max[-1.2,-0.8]} = e^{-0.8}$

$T_2(x) = e^{-1} + e^{-1}(x+1) + \frac{e^{-1}}{2}(x+1)^2$

$|R_2(x)| = \frac{e^{-0.8}}{3!}(0.2)^3$

≈ 0.000599