5.1 Vertical Asymptotes

Definition 5.1.1

When we write $\lim_{x \to a} f(x) = \infty$ we mean that there exists some N such that for any $y > N$, there exists some $\epsilon > 0$ such that $f(a \pm \epsilon) > y$.

Definition 5.1.2

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$\lim_{x \to a^-} f(x) = \infty$

$\lim_{x \to a^-} f(x) = -\infty$

$\lim_{x \to a^+} f(x) = \infty$

$\lim_{x \to a^+} f(x) = -\infty$

Example 5.1.1 Find $\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\lim_{x \to 3^-} \frac{2x}{x-3}$

Theorem 5.1.1

- $\lim_{x \to 0^+} \ln(x) = -\infty$
- $\lim_{x \to a^-} (A \tan(\omega(x-h)) + M) = \infty$ for all $a \in \{\frac{\pi}{2\omega} + h + \frac{\pi}{\omega} \cdot k, k \in \mathbb{Z}\}$ when $A > 0$
- $\lim_{x \to a^+} (A \tan(\omega(x-h)) + M) = -\infty$ for all $a \in \{\frac{\pi}{2\omega} + h + \frac{\pi}{\omega} \cdot k, k \in \mathbb{Z}\}$ when $A > 0$

Example 5.1.2 Find $\lim_{x \to 0} \ln(\tan^2(x))$.

Example 5.1.3 Find $\lim_{x \to 0^-} e^{\frac{1}{x}}$.

5.2 Horizontal Asymptotes

Definition 5.2.1

Let f be a function defined on some interval (a, ∞). Then $\lim_{x \to \infty} f(x) = L$ means that the values of $f(x)$ can be made as close to L (and *stay* as close to L) as we like by taking x sufficiently large. We call the line $y = L$ a **horizontal asymptote**.

Similarly if f is a function defined on some interval $(-\infty, b)$ then $\lim_{x \to -\infty} f(x) = L$ means that the values of $f(x)$ can be made as close to L (and *stay* as close to L) as we like by taking x sufficiently small (very negative). We still call the line $y = L$ a **horizontal asymptote**.

Example 5.2.1 Investigate $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.

Theorem 5.2.1

If n is a positive integer, then $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^n} = 0$.

Example 5.2.2 Evaluate the following limits.

a. $\lim\limits_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

b. $\lim\limits_{x\to\infty} \sin(x)$

c. $\lim\limits_{x\to\infty} (\sqrt{x^2+1} - x)$

d. $\lim\limits_{x\to\infty} (x^2 - x)$