8.1 Relating the First, Second, and Third Derivatives

In the previous Lesson we summarized the relationship between a function and its first derivative in the following table:

f	$\nearrow$	$\rightarrow$ horizontal tan	$\searrow$	$\smile$	inflection	$\frown$
f'	$+$	0	-	$\nearrow$	$\rightarrow$ horizontal tan	$\searrow$

Table 8.1.1: The relationship between f and f'.

We now want to extend this table to include the relationship between the original function and the second derivative. We start though with the following definition.

Definition 8.1.1

Given a function, f, which is differentiable on the interval (a, b) with derivative function f' also differentiable on (a, b), the **Second Derivative of** f is the derivative of the function f' on that interval.

Note that the first derivative outputs the slopes of the original function. The second derivative, being the derivative of the first derivative, then outputs the slopes of the first derivative. Thus, if the first derivative is increasing, the second derivative is positive. If the first derivative is decreasing, the second derivative is negative. If the first derivative has a horizontal tangent, then the second derivative is zero. We can summarize this in Table 8.1.2 below:

f'	$\nearrow$	$\rightarrow$ horizontal tan	$\searrow$
f''	$+$	0	-

Table 8.1.2: The relationship between f' and f''.

We then compile this information with 7.1.1 to obtain a table which relates the original function, the first derivative, and the second derivative:

f	$\nearrow$	$\rightarrow$ horizontal tan	$\searrow$	$\smile$	inflection	$\frown$	F
f'	$+$	0	-	$\nearrow$	$\rightarrow$ horizontal tan	$\searrow$	f
f''				$+$	0	-	f'

Table 8.1.3: The relationship between f, f', and f''.

Let's note the following: When the original function is concave up, this means its slopes are increasing so the first derivative is increasing, which means the second derivative is positive. Conversely, if the original function is concave down, its slopes are decreasing so the first derivative is decreasing which means the second derivative is negative. We can summarize this as:

- When the original function is concave up, the second derivative is positive.
- When the original function is concave down, the second derivative is negative.
- When the original function has an inflection point, the second derivative is zero.

Example 8.1.1 Determine which of the following graphs is f, f', f'', and f'''.

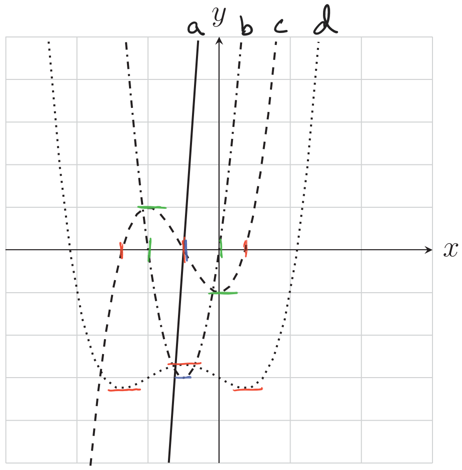


Figure 8.1.1: $y = f, f', f'', f'''$

Let's label these as a, b, c, & d.

$\frac{d}{dx}(d) = c$ c is the derivative of d.

$\frac{d}{dx}(c) = b$ b is the derivative of c.

$\frac{d}{dx}(b) = a$ a is the derivative of b

Thus $d = f$, $c = f'$, $b = f''$, & $a = f'''$.

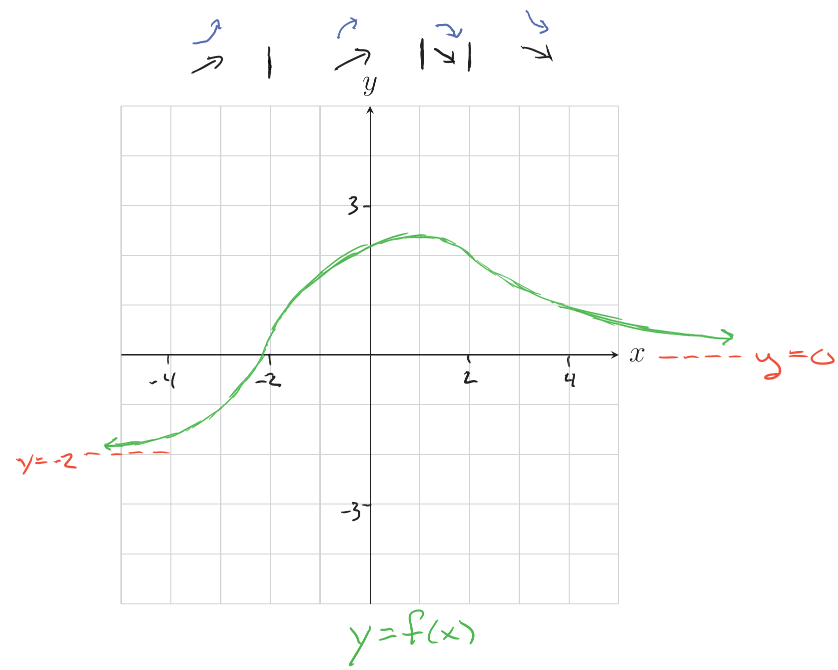
8.2 Using Information about the First and Second Derivatives in Graph Construction

Example 8.2.1 Sketch a possible graph of a function f that satisfies the following conditions:

i. $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

ii. $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

iii. $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$



8.3 Going Backwards: The Antiderivative

Example 8.3.1 Suppose the following graph is f', the derivative of a function f. Further, suppose $f(0) = 0$. Sketch a graph of f.

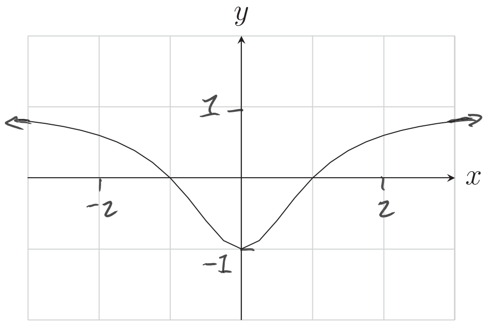
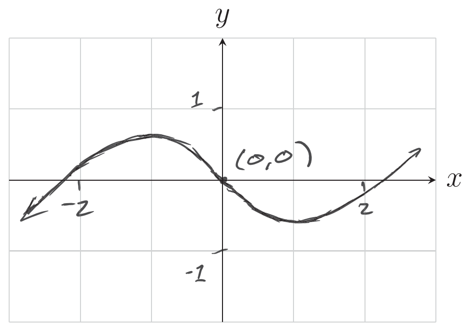


Figure 8.3.1: $y = f'(x)$



$y = f(x)$

The y-values here are the slopes of the graph we want

- $f'(-1) = f'(1) = 0$
- $f'(0) = -1$
- $f'(x)$ approaches 1 as we move to the right or left.
- symmetrical

> **Definition 8.3.1**
>
> Suppose we have a function, $y = f(x)$, then there may be a function F such that $\dfrac{d}{dx}F(x) = f(x)$. If such a function exists then we say that F is the **antiderivative** of f. Moreover, if f' is the derivative of f then we may say that f is an antiderivative of f' and similarly if f'' is the derivative of f' then f' is the antiderivative of f''.

Example 8.3.2 Let F be an antiderivative of the function f whose graph is shown in the figure below.

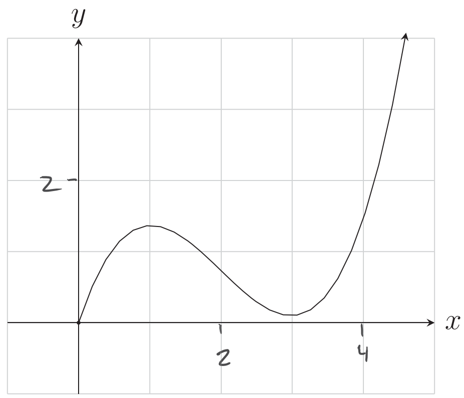


Figure 8.3.2: $y = f(x)$

a. Over what intervals is F increasing and decreasing?

The antiderivative is increasing when its derivative is positive.

increasing: $(0, \infty)$

decreasing: nowhere

b. Where is F concave upward or concave downward?

F is concave up when f is increasing.

con up: $(0,1) \cup (3, \infty)$

con dwn: $(1,3)$

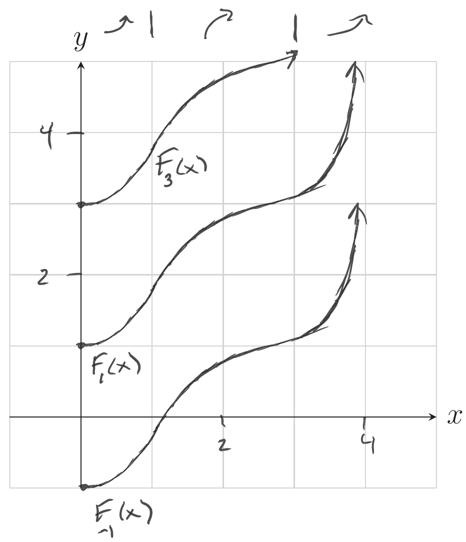
c. At what values of x does F have an inflection point?

F has an inflection when $f'(x)=0$

i.e. when f has a horizontal tangent.

So, when $x=1$ & $x=3$. This can also be seen from where the concavity changes.

d. Sketch the graph of F supposing $F(0) = 1$. Then sketch the graph of F supposing $F(0) = 2$, $F(0) = -1$, and $F(0) = 3$.



e. How many antiderivatives does f have?

An infinite number!