Math 256 LP 10 - 5.1 Matrices and Linear Systems

Solutions

1. Write the system $x' = tx - e^t y + \cos(t)$, $y' = e^{-t}x + t^2 y$ in the form $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$

$$x' = tx - e^t y$$
$$y' = e^{-t}x + t^2 y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} t & -e^t \\ e^{-t} & t^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{x}' = \begin{pmatrix} t & -e^t \\ e^{-t} & t^2 \end{pmatrix}\underline{x} \qquad \underline{f(t) = 0}$$

2. Verify that vectors $\mathbf{x}_1 = e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{x}_2 = e^{-2t}\begin{bmatrix}1\\5\end{bmatrix}$ are solution to $\mathbf{x}' = \begin{bmatrix}3 & -1\\5 & -3\end{bmatrix}\mathbf{x}$. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system.

$$\begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}\underline{x}_1 = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix} = \underline{x}_1' \quad \checkmark$$

$$\begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}\underline{x}_2 = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 5e^{-2t} \end{pmatrix} = \begin{pmatrix} -2e^{-2t} \\ -10e^{-2t} \end{pmatrix} = \underline{x}_2' \quad \checkmark$$

$$W = |\underline{x}_1 \ \underline{x}_2| = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0 \quad \text{so L.I.}$$

$$\underline{x(t)} = c_1\underline{x}_1 + c_2\underline{x}_2 = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-2t} \end{pmatrix}$$

i.e.
$$\left.\begin{aligned} a(t) &= c_1 e^{2t} + c_2 e^{-2t} \\ b(t) &= c_1 e^{2t} + 5c_2 e^{-2t} \end{aligned}\right\} \quad \text{If } \underline{x}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

3. Verify that vectors $\mathbf{x}_1 = e^{-2t}\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = e^{t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = e^{3t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are solution to $\mathbf{x}' = \begin{bmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{bmatrix}\mathbf{x}$. Then use the Wronskian to show that they are linearly independent. Write the general solution of the system and then find a particular solution of the systemwith initial conditions $x_1(0) = 5$, $x_2(0) = -7$, and $x_3(0) = 11$ $\leftarrow$ $\underline{x}(0) = \begin{pmatrix} 5 \\ -7 \\ 11 \end{pmatrix}$

$$\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\underline{x}_1 = e^{-2t}\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = e^{-2t}\begin{pmatrix} -6 \\ 4 \\ -4 \end{pmatrix} = \underline{x}_1' \quad \checkmark$$

$$\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\underline{x}_2 = e^{t}\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = e^{t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \underline{x}_2' \quad \checkmark$$

$$\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\underline{x}_3 = e^{3t}\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = e^{3t}\begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \underline{x}_3' \quad \checkmark$$

$$W = \begin{vmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{vmatrix} = 2e^{-2t}(-e^{4t} + e^{4t}) - e^{t}(-3e^{t} + 2e^{t}) = e^{2t} \neq 0 \quad \text{so L.I.}$$

$$x(t) = c_1\underline{x}_1 + c_2\underline{x}_2 + c_3\underline{x}_3$$

$$x(t) = c_1 e^{-2t}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} x_1(0) = 5 \\ x_2(0) = -7 \\ x_3(0) = 11 \end{array}\right\} \Rightarrow \underline{x}(0) = \begin{pmatrix} 5 \\ -7 \\ 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ -7 \\ 11 \end{pmatrix} = c_1\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 1 & 1 & 5 \\ -2 & -1 & -1 & -7 \\ 2 & 1 & 0 & 11 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 5/3 \\ 0 & -1/3 & -1/3 & -11/3 \\ 0 & 1/3 & -2/3 & 23/3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 11 \\ 0 & 0 & -1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -4 \end{array}\right)$$

$$x(t) = \begin{pmatrix} 3 & 1 & 1 \\ -2 & -1 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2e^{-2t} \\ 15e^{t} \\ -4e^{3t} \end{pmatrix}$$