Math 253 LP 10 - 8.8 Taylor's Inequality and Applications

1. Given the following functions, determine a value, M, such that $|f^{(n+1)}(x)| < M$ for the given value of n and the given interval.

   (a) $f(x) = x\ln(x)$, $n = 3$, $0.8 \le x \le 1.2$

   $f'(x) = \ln(x) + 1$

   $|x-1| < 0.2$

   $f''(x) = \frac{1}{x}$

   $f'''(x) = -x^{-2}$

   $f^{(4)}(x) = 2x^{-3}$

   $= \frac{2}{x^3} \le \frac{2}{0.8^3}$ on $0.8 \le x \le 1.2$

   So choose $M = \frac{2}{0.8^3}$

   Why would $M = 16$ also work

   (b) $f(x) = \sqrt[3]{x}$, $n = 1$, $7 \le x \le 9$

   $f'(x) = \frac{1}{3}x^{-2/3}$

   $f''(x) = -\frac{2}{9}x^{-4/3} = -\frac{2}{9x\sqrt[3]{x}}$

   $|f''(x)| = \frac{2}{9x\sqrt[3]{x}} \le \frac{2}{9\cdot 7\cdot\sqrt[3]{7}}$ on $7 \le x \le 9$

   on $7 \le x \le 9$

   So choose $M = \frac{2}{9\cdot 7\cdot \sqrt[3]{7}}$

   Note $|f''(x)| = \left|\frac{2}{9x^{4/3}}\right| < \left|\frac{2}{9x}\right| < \left|\frac{1}{3x}\right|$

   So choosing $M = \frac{1}{21}$ is okay also (but not as good)

2. Approximate $f(x) = \dfrac{2}{x^3}$ with T_3 about $a = 2$. Then use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when x lies in the interval $[1.5, 2.5]$. Check your results by graphing $|R_3(x)|$

$f(x) = 2x^{-3}$ $\quad f(2) = \frac{2!}{2^3} = \frac{1}{4}$

$f'(x) = -6x^{-4}$ $\quad f'(2) = -\frac{3!}{2^4} = -\frac{3}{8}$

$f''(x) = 24x^{-5}$ $\quad f''(2) = \frac{4!}{2^5} = \frac{3}{4}$

$f'''(x) = -120x^{-6}$ $\quad f'''(2) = -\frac{5!}{2^6} = -\frac{15}{8}$

$|f^{(4)}(x)| = |720x^{-7}|$

$= \left|\frac{720}{x^7}\right| \le \frac{720}{1.5^7} \approx 42.14$

Choose $M = \frac{720}{1.5^7}$

$T_3(x) = \frac{1}{4} - \frac{3}{8}(x-2) + \frac{3}{16}(x-2)^2 - \frac{5}{16}(x-2)^3$

$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

$|R_3(x)| \le \frac{720/1.5^7}{4!}|x-2|^4$

note $|x-2| \le 0.5$

$\le \frac{720}{4!\cdot 1.5^7}\cdot(.5)^4$

≈ 0.10974

3. Use Taylor's Inequality to estimate the range of values of x for which $\sin(x) \approx x - \frac{x^3}{6}$ is accurate within $|error| < 0.01$.

Note $n=3$ OR $n=4$ since

$$\sin(x) = 0 + x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^4}{24} + \dots$$

is the full expansion

Let's work with $n=3$:

Note $a=0$

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$|f^{(4)}(x)| = |\sin(x)| \leq 1$

Choose $M=1$

$$|R_3(x)| \leq \frac{1}{4!}|x|^4$$

Thus, we want

$$\frac{1}{4!}|x|^4 < 0.01$$

$$|x|^4 < 0.24$$

$$|x| < \sqrt[4]{0.24} \approx 0.6999$$