Math 253 LP 14 - 9.3 Dot Product

Solutions

Definition: The **Dot Product** of two vectors is a **scalar**, denoted by $\mathbf{a} \cdot \mathbf{b}$, which is computed via:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + ... a_n b_n$$

where n is the dimension of the vectors. The vectors must be of the same dimension in order to take the dot product.

The dot product may also be calculated via:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

where θ is the angle between the vectors. Note $0 \leq \theta \leq \pi$.

1. What is $\langle 2, -1, 3 \rangle \cdot \langle 3, 2, -5 \rangle$?

   $\langle 2, -1, 3 \rangle \cdot \langle 3, 2, -5 \rangle = 6 - 2 - 15 = -19$

2. What is the angle between the vectors $\mathbf{u} = \langle 1, 3, 2 \rangle$ and $\mathbf{v} = \langle -2, 2, 3 \rangle$?

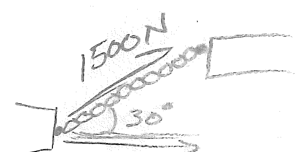
   $\cos(\theta) = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| \cdot |\underline{v}|} = \frac{10}{\sqrt{14} \cdot \sqrt{17}} \Rightarrow \theta = \cos^{-1}\left(\frac{10}{\sqrt{14} \cdot \sqrt{17}}\right) \approx 49.6^\circ$

3. Suppose $\mathbf{a} \cdot \mathbf{b} = 0$. What is the angle between the vectors? Conversely, if two vectors are orthogonal, what is the dot product of the vectors?

   Well, if $\underline{a}$ or $\underline{b}$ is zero, the question is meaningless but otherwise the angle is 90°. Thus $\underline{a} \cdot \underline{b} = 0$ means either 90° or one of them is zero. If the angle is 90° then $\underline{a} \cdot \underline{b} = 0$ always.

4. A tow truck drags a stalled car along a road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N. Use the dot product to determine how much work is done by the truck in pulling the car 1 km?

   $J = N \cdot m$



$|F_P| = 1500\cos(30^\circ)$

$W = |F_P| \cdot D$

$= 1500\cos(30^\circ) \cdot 1000$

$= 1299.038$ kJ

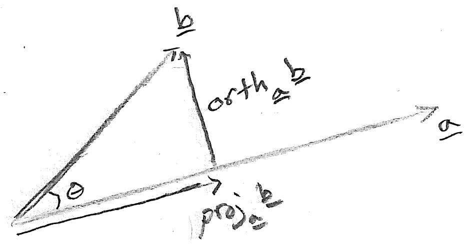
In lbs $\qquad W = F \cdot D$

5. A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P = (2,1,0)$ to the point $Q = (4,6,2)$. Find the work done. Distance in feet

$$W = \langle 3,4,5\rangle \cdot \langle 2,5,2\rangle$$

$$= 6 + 20 + 10 = 36 \text{ ft-lbs.}$$

6. Given vectors $\mathbf{b}$ and $\mathbf{a}$, determine a formula which describes the vector which is the projection of $\mathbf{b}$ onto the vector $\mathbf{a}$ using a dot product. What is the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$?



$$|\text{proj}_{\underline{a}}\underline{b}| = |\underline{b}|\cos(\theta)$$

$$= |\underline{b}| \cdot \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|\,|\underline{b}|} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|}$$

$$\text{proj}_{\underline{a}}\underline{b} = |\text{proj}_{\underline{a}}\underline{b}| \cdot \frac{\underline{a}}{|\underline{a}|} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\,\underline{a}$$

$$\text{orth}_{\underline{a}}\underline{b} = \underline{b} - \text{proj}_{\underline{a}}\underline{b}$$

7. Determine the projection of $\mathbf{b}$ onto $\mathbf{a}$ for the given vectors and then determine $\text{orth}_{\mathbf{a}}\mathbf{b}$.

$\mathbf{b} = \langle 1,1,2,\rangle$, $\mathbf{a} = \langle -2,3,1\rangle$

$$\text{proj}_{\underline{a}}\underline{b} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\underline{a} = \frac{3}{14}\langle -2,3,1\rangle = \langle -\tfrac{3}{7}, \tfrac{9}{14}, \tfrac{3}{14}\rangle$$

$$\text{orth}_{\underline{a}}\underline{b} = \underline{b} - \text{proj}_{\underline{a}}\underline{b} = \langle 1,1,2\rangle - \langle -\tfrac{3}{7}, \tfrac{9}{14}, \tfrac{3}{14}\rangle = \langle \tfrac{10}{7}, \tfrac{5}{14}, \tfrac{25}{14}\rangle$$

8. Given vectors $\mathbf{a} = \langle 2,1\rangle$ and $\mathbf{b} = \langle -1,3\rangle$, set up an ortho-normal base using $\mathbf{a}$ and $\text{orth}_{\mathbf{a}}\mathbf{b}$ and then describe $\mathbf{b}$ in terms of this base.

$$\text{proj}_{\underline{a}}\underline{b} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\underline{a} = \frac{1}{5}\langle 2,1\rangle = \langle \tfrac{2}{5}, \tfrac{1}{5}\rangle$$

$$\text{orth}_{\underline{a}}\underline{b} = \langle -1,3\rangle - \langle \tfrac{2}{5}, \tfrac{1}{5}\rangle = \langle -\tfrac{7}{5}, \tfrac{14}{5}\rangle$$

$$\underline{b} = \frac{1}{5}\underline{a} + \text{orth}_{\underline{a}}\underline{b}$$

Let $\beta = \{\langle 2,1\rangle, \langle -\tfrac{7}{5}, \tfrac{14}{5}\rangle\}$

$$= \langle \tfrac{1}{5}, 1\rangle_\beta$$