Math 253 LP 14 - 9.4 Cross Product

Solutions

Definition: The **cross product** (sometimes called the **vector product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$, notated $\mathbf{a} \times \mathbf{b}$, results in the *vector* which is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. The length of this resultant vector is $|\mathbf{a}||\mathbf{b}|\sin(\theta)$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ with $0 \leq \theta \leq \pi$. The direction of the resultant vector points out of the white board if you move counterclockwise to form the angle from $\mathbf{a}$ to $\mathbf{b}$ and into the white board if you move clockwise to form the angle from $\mathbf{a}$ to $\mathbf{b}$.

1. In general, given two vectors $\mathbf{a}$ and $\mathbf{b}$, how does $\mathbf{a} \times \mathbf{b}$ relate to $\mathbf{b} \times \mathbf{a}$?

This changes the angle between the vectors from/to clockwise to/from counterclockwise so $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

Thus the cross product is NOT commutative.

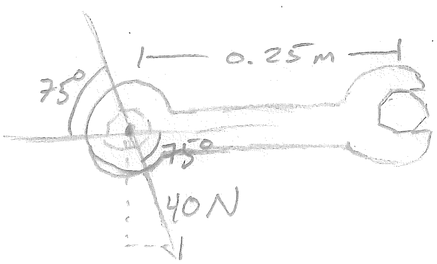
2. Find $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{i} \times \mathbf{j})$

$$(\vec{i} \times \vec{i}) \times \vec{j} = (\vec{0}) \times \vec{j} = \vec{0}$$

$$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$$

Thus the cross product is NOT associative

3. A bolt is tightened by applying a 40-N force to a 0.25-m wrench at an angle of 75° with the arm of the wrench. Find the magnitude of the torque about the center of the bolt.



$$|\tau| = |l \cdot F_\perp|$$

$$= \underbrace{0.25 \cdot 40\sin(75^\circ)}_{|\underline{l} \times \underline{F}|} \approx 9.66$$

The magnitude of the torque is about 9.66 Nm

4. Calculate the cross products of the following:

a. $\mathbf{a} = \langle 1, 3, 4 \rangle$, $\mathbf{b} = \langle 2, 7, -5 \rangle$

$$\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 7 \\ -5 \end{pmatrix} = \langle -43, 13, 1 \rangle$$

b. $\mathbf{a} = \langle -2, 3, 5 \rangle$, $\mathbf{b} = \langle -1, -1, 0 \rangle$

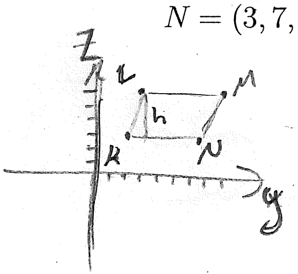
$$\begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \langle 5, 5, 5 \rangle$$

5. Find a vector perpendicular to the plane that passes through the points $P = (1, 4, 6)$, $Q = (-2, 5, -1)$, and $R = (1, -1, 1)$.

$\vec{PQ} = \langle -3, 1, -7 \rangle \qquad \vec{PR} = \langle 0, -5, -5 \rangle$

$$\begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \times \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix} = \langle -40, -15, 15 \rangle$$

6. Find the area of the parallelogram with vertices $K = (1, 2, 3)$, $L = (1, 3, 6)$, $M = (3, 8, 6)$, and $N = (3, 7, 3)$.



$\vec{KL} = \langle 0, 1, 3 \rangle \qquad \vec{MN} = \langle 0, -1, -3 \rangle$ parallel

$\vec{KN} = \langle 2, 5, 0 \rangle$

$$\left| \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} \right| = |\langle -15, 6, -2 \rangle|$$

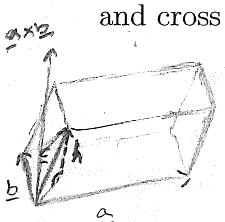
$$= \sqrt{225 + 36 + 4}$$

$$= \sqrt{265}$$

$A = |\vec{KN}| \cdot h$

$= |\vec{KN}| \cdot |\vec{KL}| \sin(\theta) = |\vec{KN} \times \vec{KL}|$

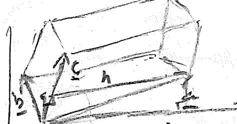
7. Determine the volume of a parallelepiped, described by vectors **a**, **b**, and **c**, in terms of dot and cross products in two different manners.



$$V = |A_{|a \times b|} \cdot h| = \left| |a \times b| \cdot \frac{(a \times b) \cdot c}{|a \times b|} \right| = |(a \times b) \cdot c|$$

$h = |\text{proj}_{a \times b}\, c| = |c| \cos\theta$

$= (a \times b) \cdot c \,/\, |a \times b|$



$V = |A_{|c \times b|} \cdot h|$

$= |a \cdot (b \times c)|$

$h = |\text{proj}_{b \times c}\, a|$

$$= \frac{(b \times c) \cdot a}{|b \times c|}$$

8. Determine the volume of the parallelepiped, described by the vectors $\mathbf{a} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$(a \times b) \cdot c = \left| \left[\begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right] \cdot \langle -2, 2, 1 \rangle \right|$$

$$= |\langle -4, -10, 3 \rangle \cdot \langle -2, 2, 1 \rangle| = |8 - 20 + 3| = 9$$