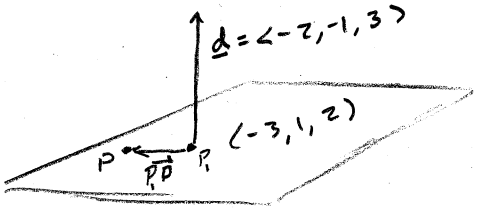
Math 253 LP 16 - 9.5b Equations of Planes

1. Find the equation of the plane through the point $(-3, 1, 2)$ and perpendicular to the vector $\langle -2, -1, 3\rangle$.



If P is any other point on the plane with $P = (x, y, z)$, then $\overrightarrow{P_1P} \cdot \underline{n} = 0$ since $\underline{n}$ is $\perp$ to the plane. (We call $\underline{d}$ a normal vector)

If $\underline{n} = \langle a, b, c\rangle$ & $P_1 = (x_1, y_1, z_1)$

$\overrightarrow{P_1P} \cdot \underline{n} = 0$

$\Rightarrow \langle x - x_1, y - y_1, z - z_1\rangle \cdot \langle a, b, c\rangle = 0$

$\Rightarrow ax + by + cz - ax_1 - by_1 - cz_1 = 0$

$\Rightarrow ax + by + cz = d$ where $d = ax_1 + by_1 + cz_1$

In practice: $-2x - y + 3z = d$ Put $\underline{n}$ in for a, b, c

Plug in P_1: $\underset{11 = d}{-2(-3) - (1) + 3(2) = d}$ $\Rightarrow$ Plane is $-2x - y + 3z = 11$

2. Determine the equation of the plane which contains the points $\underset{A}{(0, 1, 1)}$, $\underset{B}{(1, 0, 1)}$, and $\underset{C}{(1, 1, 0)}$.

We need a normal vector:

$\underline{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \langle 1, 1, 1\rangle$

$x + y + z = d$

Use $(0,1,1)$: $0 + 1 + 1 = d$

$x + y + z = 2$

3. Are the following planes parallel, perpendicular, or neither? If neither, what is the angle between them?
$x + 2y + 2z = 1 \quad 2x - y + 2z = 1$

$\underline{n_1} = \langle 1, 2, 2\rangle$ $\quad \underline{n_2} = \langle 2, -1, 2\rangle$ Not parallel ($\underline{n_1} \neq k\underline{n_2}$ for any k)

$\underline{n_1} \cdot \underline{n_2} = 2 - 2 + 4 = 4$ Not perp.

$\Rightarrow$ Neither $\quad \cos(\theta) = \dfrac{\underline{n_1} \cdot \underline{n_2}}{|\underline{n_1}|\,|\underline{n_2}|} = \dfrac{4}{3 \cdot 3} = \dfrac{4}{9}$

$\theta = \cos^{-1}(4/9) \approx 63.6^\circ$

4. Where does the line through $(2, -3, 1)$ and $(3, -4, 2)$ intersect the plane $2x - y + 3z = 6$?

ℓ: $\underline{r} = \langle 2,-3,1\rangle + t\langle 1,-1,1\rangle \qquad x = 2+t \quad y = -3-t \quad z = 1+t$

$2(2+t) - (-3-t) + 3(1+t) = 6 \qquad x = 4/3 \quad y = -7/3 \quad z = 1/3$

$10 + 6t = 6$

$t = -2/3 \qquad P = (4/3, -7/3, 1/3)$

5. What is the equation of the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$? We need a point & a direction for $\underline{r} = \underline{p} + t\underline{d}$

If the line goes through the xy-plane we can find the point where it does by setting $z=0$:

$x + 2y = 1$

$-(x - y = 1) \qquad x = 1$

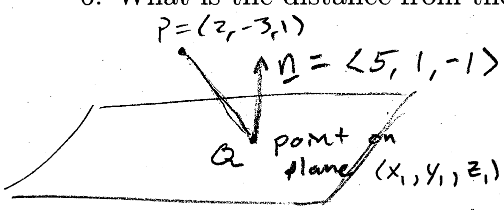
$3y = 0 \Rightarrow y = 0 \qquad$ So $P = (1,0,0)$

The direction is perp. to both normal vectors so:

$\underline{d} = \begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix} = \langle 5, 2, -3\rangle$

Thus $\underline{r} = \langle 1,0,0\rangle + t\langle 5,2,-$

6. What is the distance from the point $(2, -3, 1)$ to the plane given by $5x + y - z = 1$?



Q on plane so $5x_1 + y_1 - z_1 = 1$

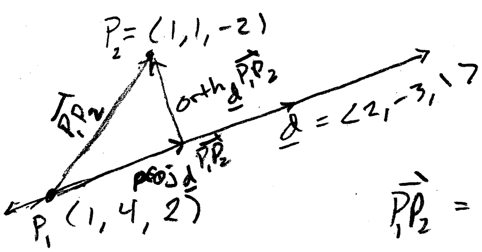
$$D = \left|\text{comp}_{\underline{n}}\,\overrightarrow{QP}\right| = \left|\frac{\overrightarrow{QP}\cdot\underline{n}}{|\underline{n}|}\right| = \left|\frac{\langle 2-x_1, -3-y_1, 1-z_1\rangle\cdot\langle 5,1,-1\rangle}{\sqrt{27}}\right|$$

$$= \frac{|\langle 2,-3,1\rangle\cdot\langle 5,1,-1\rangle - \langle x_1, y_1, z_1\rangle\cdot\langle 5,1,-1\rangle|}{\sqrt{27}}$$

$$= \frac{6-1}{\sqrt{27}} = 5\sqrt{27}/27$$

More Generally: $D = \dfrac{|ap_1 + bp_2 + cp_3 - d|}{\sqrt{a^2+b^2+c^2}}$ where $P = (p_1, p_2, p_3)$

7. What is the distance between the point $(1, 1, -2)$ to the line given by $x = 1 + 2s$, $y = 4 - 3s$, and $z = 2 + s$?



$\overrightarrow{P_1P_2} = \langle 0, -3, -4\rangle$

$$\text{proj}_{\underline{d}}\overrightarrow{P_1P_2} = \frac{\overrightarrow{P_1P_2}\cdot\underline{d}}{|\underline{d}|^2}\underline{d} = \frac{\langle 0,-3,-4\rangle\cdot\langle 2,-3,1\rangle}{4+9+1}\langle 2,-3,1\rangle$$

$$= \frac{5}{14}\langle 2,-3,1\rangle = \left\langle \frac{5}{7}, \frac{-15}{14}, \frac{5}{14}\right\rangle$$

$$\text{orth}_{\underline{d}}\overrightarrow{P_1P_2} = \overrightarrow{P_1P_2} - \text{proj}_{\underline{d}}\overrightarrow{P_1P_2} = \langle 0,-3,-4\rangle - \left\langle \frac{5}{7}, \frac{-15}{14}, \frac{5}{14}\right\rangle$$

$$= \left\langle \frac{-5}{7}, \frac{-27}{14}, \frac{-61}{14}\right\rangle$$

$$D = \left|\left\langle \frac{-5}{7}, \frac{-27}{14}, \frac{-61}{14}\right\rangle\right| = \sqrt{\frac{25}{49} + \frac{27^2}{196} + \frac{61^2}{196}}$$

$$= \frac{5\sqrt{182}}{14}$$