Math 253 LP 1 - 8.1 Sequences

Solutions

1. Determine a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{-\frac{1}{4}, \frac{4}{8}, -\frac{9}{16}, \frac{16}{32}, \ldots\right\}$$

Note the alternating sign which can be modeled by $(-1)^n$

The numerators are perfect squares so n^2 should work.

The denominators are powers of 2 but start at 2^2 so 2^{n+1}

$$a_n = (-1)^n \cdot \frac{n^2}{2^{n+1}}$$

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

a. $a_n = \dfrac{2n^2}{n^2+3}$

grow at same rate due to same degree

$$\lim_{n\to\infty} \frac{2n^2}{n^2+3} = 2$$

(by ratio of $2n^2/n^2$)

c. $a_n = \dfrac{n^3}{e^{n^2}}$

$$\lim_{n\to\infty} \frac{n^3}{e^{n^2}} = 0$$

by exponentials growing faster than polynomials

b. $a_n = \dfrac{\cos(n)}{n^2}$

$$-1 \le \cos(n) \le 1$$

$$-\frac{1}{n^2} \le \frac{\cos(n)}{n^2} \le \frac{1}{n^2}$$

$$\lim_{n\to\infty} -\frac{1}{n^2} \le \lim_{n\to\infty} \frac{\cos(n)}{n^2} \le \lim_{n\to\infty} \frac{1}{n^2}$$

$$0 \le \lim_{n\to\infty} \frac{\cos(n)}{n^2} \le 0$$

$$\Rightarrow a_n \to 0$$

d. $a_n = \dfrac{(-2)^n}{(n+1)!}$

ignoring the negative:

$$\frac{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}{(n+1) \cdot n \cdot (n-1) \cdots 3 \cdot 2} \quad \leftarrow n \text{ times}$$

$$\le \frac{2}{n+1} \to 0$$

So $a_n \to 0$ since $\pm 0 = 0$

Note: Factorials grow "faster" than exponentials

3. Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess.

a. $a_n = \sqrt[n]{3^n + 5^n}$

See Desmos

Use L'Hospital

$$\lim_{n\to\infty} (3^n + 5^n)^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(3^n+5^n)}$$

$$= \lim_{n\to\infty} e^{\frac{\ln(3)\cdot 3^n + \ln(5)5^n}{3^n+5^n} \cdot \frac{1/5^n}{1/5^n}}$$

$$= \lim_{n\to\infty} e^{\frac{\ln(3)\cdot(\frac{3}{5})^n + \ln(5)}{(3/5)^n + 1}}$$

$$= e^{\ln(5)}$$

$$= 5$$

b. $a_n = \dfrac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{(2n)^n} = \dfrac{\prod_{i=1}^{n}(2i)}{(2n)^n}$

$$= \frac{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}{2n\cdot 2n\cdot 2n \cdots 2n}$$

$$= \frac{1\cdot 2\cdot 3\cdot \cdots \cdot n}{n\cdot n\cdot n \cdots n}$$

$$\leq \frac{1}{n} \to 0$$

See Desmos

4. Explore the sequence given by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-2} + f_{n-1}$.

$f_1 = 1$

$f_2 = 1$

$f_3 = 1 + 1 = 2$

$f_4 = 1 + 2 = 3$

$f_5 = 2 + 3 = 5$

$f_6 = 3 + 5 = 8$

$f_7 = 5 + 8 = 13$

$f_8 = 8 + 13 = 21$

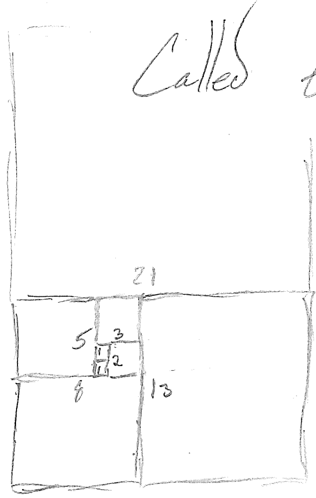
$f_9 = 13 + 21 = 34$

$f_{10} = 55$

$f_{11} = 89$

$f_{12} = 144$

Called the Fibonacci Sequence



Approximates Golden Rectangle

$$\lim_{n\to\infty} f_n = \Phi \leftarrow \text{Golden Ratio}$$

$$= \frac{1+\sqrt{5}}{2}$$

Which occurs iff $\dfrac{a+b}{a} = \dfrac{a}{b}$