Math 256 LP 3 - 1.5 and 1.6 Linear First Order Equations and Substitution Method

1. Find general solutions of the following differential equations.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

a. $(x^2+1)\frac{dy}{dx} + 3xy = 6x$

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$\rho(x) = e^{\int \frac{3x}{x^2+1}dx} \qquad u = x^2+1 \quad du = 2x$$

$$= e^{\frac{3}{2}\int \frac{du}{u}}$$

$$= e^{\frac{3}{2}\ln|u|}$$

$$= \left(e^{\ln|u|}\right)^{3/2}$$

$$= |u|^{3/2} = (x^2+1)^{3/2}$$

$$(x^2+1)^{3/2}\frac{dy}{dx} + (x^2+1)^{1/2}\cdot 3xy = (x^2+1)^{1/2}\cdot 6x$$

$$D_x\left[(x^2+1)^{3/2}y\right] = 6x(x^2+1)^{1/2}$$

$$(x^2+1)^{3/2}\cdot y = 6\int x(x^2+1)^{1/2}dx \qquad u = x^2+1 \quad du = 2x\,dx$$

$$(x^2+1)^{3/2}\cdot y = 3\int u^{1/2}du$$

$$(x^2+1)^{3/2}\cdot y = 2u^{3/2} + C$$

$$y = 2 + \frac{C}{(x^2+1)^{3/2}}$$

OR: As above, $e^{\int P(x)dx} = e^{\int \frac{3x}{x^2+1}dx} = (x^2+1)^{3/2}$

$$y(x) = \frac{1}{(x^2+1)^{3/2}}\int \frac{6x}{x^2+1}(x^2+1)^{3/2}dx$$

$$= \frac{1}{(x^2+1)^{3/2}}\cdot\int 6x(x^2+1)^{1/2}dx \qquad u = x^2+1 \quad du = 2x\,dx$$

$$= \frac{1}{(x^2+1)^{3/2}}\cdot 3\int u^{1/2}du$$

$$= \frac{3}{(x^2+1)^{3/2}}\left(\frac{2}{3}u^{3/2} + C\right)$$

$$= 2 + \frac{C_2}{(x^2+1)^{3/2}}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$e^{\int P(x)dx}\cdot\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

$$D_x\left[e^{\int P(x)dx}\cdot y(x)\right] = Q(x)\cdot e^{\int P(x)dx}$$

$$e^{\int P(x)dx}\cdot y(x) = \int Q(x)\cdot e^{\int P(x)dx}dx$$

$$y(x) = e^{-\int P(x)dx}\int Q(x)\cdot e^{\int P(x)dx}dx$$

Note IF $\rho(x) = e^{\int_{x_0}^{x} P(t)\,dt}$

$y(x) = \frac{1}{\rho(x)}\left[y_0 + \int_{x_0}^{x} \rho(t)Q(t)\,dt\right]$

Then $y(x_0) = y_0$

b. $x^2 \frac{dy}{dx} + xy = \sin(x)$ $\qquad y(1) = y_0$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin(x)}{x^2}$$

$$\rho(t) = e^{\int_1^x \frac{1}{t}dt} = e^{\ln(t)} = t$$

$$y(x) = \frac{1}{x}\int_1^x \frac{\sin(t)}{t^2}\cdot t\,dt$$

$$= \frac{1}{x}\int_1^x \frac{\sin(t)}{t}\,dt$$

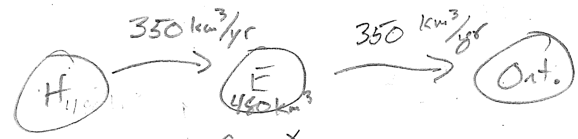
$$= \frac{1}{x}\left(\int_0^x \frac{\sin(t)}{t}\,dt - \int_0^1 \frac{\sin(t)}{t}\,dt\right)$$

$$= \frac{1}{x}\left(Si(x) - Si(1)\right)$$

Note $\int_0^x \frac{\sin(t)}{t}\,dt := Si(x)$ is not explicitly solvable (approximations can be made)

2. Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake ~~Huron~~ Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$, the pollutant concentration of Lake Erie is five times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

$$\frac{dx}{dt} = 350 \cdot C_H - 350 \cdot \frac{x}{480}$$



C_H is constant $\qquad C_E = \frac{x}{480} \qquad C_{E_0} = 5C_H$

$$\frac{dx}{dt} + \frac{35}{48}x = 350\,C_H \qquad \rho(x) = e^{\int \frac{35}{48}dx} = e^{\frac{35}{48}t}$$

Let x be the volume of pollutants in Lake Erie

Assume C_H is constant

Then $x_0 = 480 \cdot C_E = 480 \cdot 5 \cdot C_H = 2400\,C_H$

$$x(t) = e^{-\frac{35}{48}t}\cdot\int 350\cdot C_H \cdot e^{\frac{35}{48}t}\,dt$$

$$= e^{-\frac{35}{48}t}\cdot\left(350\cdot C_H\cdot\frac{48}{35}e^{\frac{35}{48}t} + A\right)$$

$$= 480\,C_H + Ae^{-\frac{35}{48}t}$$

$$2400\,C_H = 480\,C_H + A$$

$$A = 1920\,C_H$$

$$x(t) = 480\,C_H + 1920\,C_H e^{-\frac{35}{48}t}$$

$$C_E(t) = C_H + 4C_H e^{-\frac{35}{48}t}$$

When is $C_E = 2C_H$? $\qquad C_E = \frac{x}{480}$

$$2C_H = C_H + 4C_H e^{-\frac{35}{48}t}$$

$$C_H = 4C_H e^{-\frac{35}{48}t}$$

$$\frac{1}{4} = e^{-35/48\,t}$$

$$\ln(1/4) = -\frac{35}{48}t$$

$$t = -\frac{48}{35}\ln(1/4) \approx 1.9$$

It will take about 1.9 years

3. Solve the non-linear differential equation $2xyy' = x^2 + 2y^2$ using the substitution $\nu = \frac{y}{x}$

$y' = \frac{1}{2}\frac{x}{y} + \frac{y}{x}$ $\quad v = y/x \quad$ $y = xv$

$\frac{dy}{dx} = \frac{1}{2}v^{-1} + v$ $\qquad$ $\frac{dy}{\delta x} = v + x \cdot \frac{\delta v}{\delta x}$

$\frac{1}{2}v^{-1} + v = v + x\frac{dv}{\delta x}$

$\frac{1}{2}v^{-1} = x\frac{dv}{dx}$

$\int \frac{1}{x}dx = \int 2v\,dv$

$\ln|x| = v^2 + C$

$\ln|x| = \frac{y^2}{x^2} + C$

$x^2 \ln|x| - Cx^2 = y^2$

$y^2 = x^2(\ln|x| - C)$