Math 256 LP 4 - 2.1 and 2.2 Population Models and Equilibrium Solutions

1. Solve the initial value problem $\frac{dx}{dt} = 10x - x^2$, $x(0) = 1$. Then sketch the graphs of several solutions of the given differential equation and highlight the indicated particular solution.

$$\frac{dx}{dt} = 10x - x^2$$

$$\int \frac{1}{x(10-x)}\,dx = \int dt$$

$$\int \frac{1}{10x} + \frac{1}{10(10-x)}\,dx = t + C_1$$

$$\frac{1}{10}\left(\ln|x| - \ln|10-x|\right) = t + C_1$$

$$\ln\left|\frac{x}{10-x}\right| = 10t + C_2$$

$$\frac{x}{10-x} = Ae^{10t}$$

$$x = 10Ae^{10t} - xAe^{10t}$$

$$\frac{1}{x(10-x)} = \frac{A}{x} + \frac{B}{10-x}$$

$$1 = A(10-x) + Bx$$

$$1 = 10A + (B-A)x$$

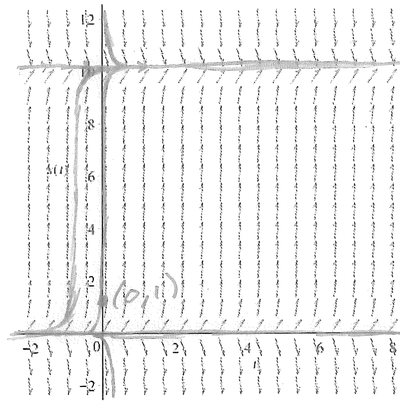
$10A = 1$, $A = 1/10$ $\qquad$ $B - A = 0$, $B = A = 1/10$

$$x + xAe^{10t} = 10Ae^{10t}$$

$$x(1 + Ae^{10t}) = 10Ae^{10t}$$

$$x(t) = \frac{10Ae^{10t}}{1 + Ae^{10t}}$$

$$1 = \frac{10A}{1+A} \Rightarrow 1 + A = 10A \Rightarrow A = 1/9$$



$x(t) = 10$

$x(t) = 0$

$$x(t) = \frac{10e^{10t}}{9 + 10e^{10t}}$$

2. Suppose that when a certain lake is stocked with fish, the birth and death rates, β and δ are both inversely proportional to $\sqrt{P}$. Show that $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$ for some constant k. If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

$$\frac{dP}{dt} = (\beta - \delta)P = \left(\frac{k_1}{\sqrt{P}} - \frac{k_2}{\sqrt{P}}\right)P = P^{\frac{1}{2}}(k_1 - k_2) = P^{1/2}k$$

$$\Rightarrow \quad P^{-1/2}\,dP = k\,dt$$

$$2P^{1/2} = kt + C \qquad P(0) = P_0$$

$$2\sqrt{P_0} = C$$

$$2P^{1/2} = kt + 2\sqrt{P_0}$$

$$P^{1/2} = \frac{1}{2}kt + \sqrt{P_0}$$

a) $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$

b) $P(t) = \left(\frac{1}{2}kt + 10\right)^2$

$169 = \left(\frac{1}{2}k \cdot 6 + 10\right)^2$ $\qquad$ t in months

$13 = 3k + 10$

$k = 1$

$P(t) = \left(\frac{1}{2}t + 10\right)^2$

$P(12) = 16^2 = 196$

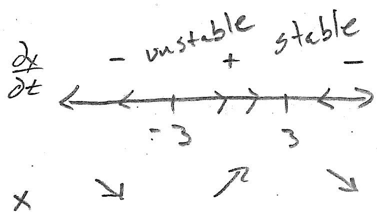
There will be 196 fish after 1 year.

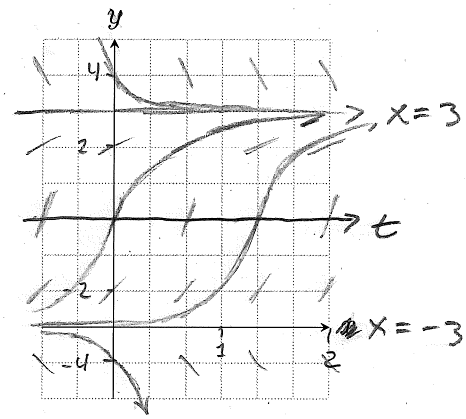
3. Given $\frac{dx}{dt} = f(x)$, first solve the equation $f(x) = 0$ to find the critical points of the given autonomous differential equation. Then analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for $x(t)$ in terms of t. Finally, sketch typical solution curves for the given differential equation and verify visually the stability of each critical point.

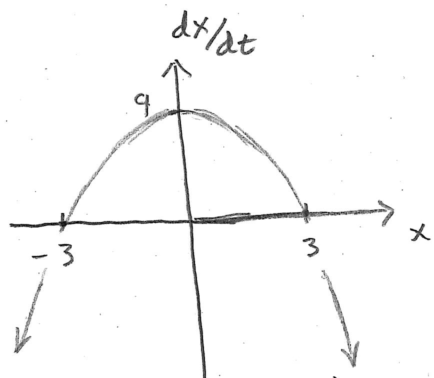
$\frac{dx}{dt} = 9 - x^2$

$0 = 9 - x^2$

$x = \pm 3$







$\frac{dx}{dt} = 9 - x^2$

$\int \frac{1}{9-x^2}\,dx = \int dt$

$\frac{1}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}$

$\left(\int \frac{1/6}{3-x} + \frac{1/6}{3+x}\,dx = t + c_1\right)\cdot 6$

$1 = A(3+x) + B(3-x)$

$1 = (3A+3B) + (A-B)x$

$-\ln|3-x| + \ln|3+x| = 6t + C_2$

$3A + 3B = 1$ $\quad A = B$

$6B = 1$

$B = 1/6 = A$

$\ln\left|\frac{3+x}{3-x}\right| = 6t + C_2$

$\frac{3+x}{3-x} = Ae^{6t}$

$3 + x = 3Ae^{6t} - xAe^{6t}$

$x + xAe^{6t} = 3Ae^{6t} - 3$

$x(t) = \frac{3Ae^{6t} - 3}{1 + Ae^{6t}}$