Math 253 LP 5 - 8.4 Other Convergence Tests

Solutions

1. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ for convergence or divergence.

$$\lim_{n\to\infty} \left|\frac{(-1)^{n-1}}{2n-1}\right| = \lim_{n\to\infty} \frac{1}{2n-1} = 0$$

So the series is convergent.

By the alternating series test, Given $\sum a_n$ where the a_n are alternating, it is convergent iff $\lim_{n\to\infty} |a_n| = 0$

2. Given the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 5^n}$, show that the series is convergent then find the sum with $|\text{error}| < 0.0005$.

By Alternating Series Estimation Theorem, Given $\sum_{i=1}^{\infty} a_i$ (alternating):

$$\left|\text{error}\left(\sum_{i=1}^{n} a_i\right)\right| < |a_{n+1}|$$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n \cdot 5^n}\right| = \lim_{n\to\infty} \frac{1}{n \cdot 5^n} = 0$$

So convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 5^n} = -\frac{1}{5} + \frac{1}{2\cdot 5^2} - \frac{1}{3\cdot 5^3} + \frac{1}{4\cdot 5^4} - \frac{1}{5\cdot 5^5} + \ldots \approx -0.182330\overline{6}$$

0.2   0.02 $0.002\overline{6}$ 0.0004   0.000064 ✓

Use 1st 4 terms to get

3. Determine whether the following series are absolutely convergent or not.

a. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

Ratio Test: Given $\sum_i b_n$

Take $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|$

If you get a real #.

$L < 1$, then $\sum_i b_n$ is absolutely convergent ($\Rightarrow$ convergent)

If $L > 1$ or $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \infty$ then $\sum_i b_n$ is divergent.

If $L = 1$ it is inconclusive. Find a different test.

$$\lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{\cancel{3^n}}{\cancel{3^n}\cdot 3}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}$$

$$= \frac{1}{3}$$

Thus the series is absolutely convergent ($\Rightarrow$ convergent).

Note the AST would show convergence but not absolute convergence.

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(n+1)!} \cdot \frac{n!}{n^n}$$

Thus the series is divergent.

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \frac{(n+1)^n(n+1)}{n^n}$$

$$= \lim_{n\to\infty} \frac{n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n+1)(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1} \cdot \frac{n+1}{1} \cdot \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} 1\cdot\left(1+\frac{1}{n}\right)^n = e$$

c. $\sum_{n=1}^{\infty} \frac{\cos(3n)}{3^n}$

$$\sum_{n=1}^{\infty} \left|\frac{\cos(3n)}{3^n}\right| < \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1/3}{1-1/3} = \frac{1}{2}$$

+ is thus absolutely convergent.