Math 253 LP 6 - 8.5 Power Series

Solutions

1. Graph the function $f(x) = \dfrac{1}{1-x}$ and the power series $P(x) = \sum_{n=0}^{\infty} x^n$ using the first k-terms where you start with $k = 1$ and up that number until you have a good idea of what the radius of convergence and interval of convergence are. Then find the radius of convergence and interval of convergence of $P(x)$ symbolically.

See Desmos

$P(x) = \sum_{n=0}^{\infty} x^n$ is geometric so converges on $(-1,1)$ with $R=1$

Or, via Ratio Test:

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|$

$|x| < 1$ when $-1 < x < 1$

Then Endpoints:

$x=-1$: $\sum_{n=0}^{\infty} (-1)^n$ diverges by divergence test

$x=1$: $\sum_{n=0}^{\infty} (1)^n$ diverges by divergence test

2. Find the radius of convergence and interval of convergence for the following series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n^3}$

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{x^n} \right| = \lim_{n\to\infty} \left| x \cdot \left(\frac{n}{n+1}\right)^3 \right|$

$= |x \cdot 1| = |x|$

$|x| < 1$ when $-1 < x < 1$

Ends: $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^3}$

$= -\sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. by p-test

$x=1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ conv by A.S.T.

$I = [-1, 1]$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{2^{2n+2} \cdot ((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{2^2 \cdot (n+1)^2} \right| = 0$

Hence $I = (-\infty, \infty)$

3. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geq 0$, find the interval of convergence of the series and a formula for $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_0 x^4 + c_1 x^5 + c_2 x^6 + \cdots$$

$$= c_0(1 + x^4 + x^8 + x^{12} + \cdots) + c_1(x + x^5 + x^9 + \cdots) + c_2(x^2 + x^6 + \cdots) + c_3(x^3 + x^7 + \cdots)$$

$$= c_0(1 + x^4 + x^8 + \cdots) + c_1 x(1 + x^4 + x^8 + \cdots) + c_2 x^2(1 + x^4 + x^8 + \cdots) + c_3 x^3(1 + x^4 + x^8 + \cdots)$$

$$= (c_0 + c_1 x + c_2 x^2 + c_3 x^3)\underbrace{(1 + x^4 + x^8 + \cdots)}$$

Geometric with $a = 1$, $r = x^4$

$$= (c_0 + c_1 x + c_2 x^2 + c_3 x^3) \cdot \frac{1}{1 - x^4}$$

Converges when $-1 < x^4 < 1$

$\Rightarrow -1 < x < 1$

$I = (-1, 1)$

Wait! There's a special case!

If $c_0 = c_1 = c_2 = c_3 = 0$ then

$f(x) = 0$ + converges on $\mathbb{R}$.