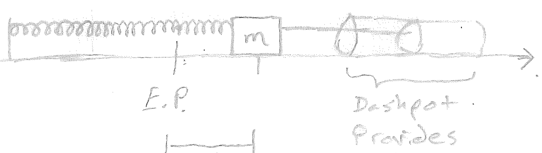
Math 256 LP 7 - 3.4 and 3.5 Mechanical Vibrations and Nonhomogeneous Equations

Name: Solutions

1. Determine a 2nd-order differential equation describing the motion of a mass-spring-dashpot system. Find a general solution to this system and then find the specific position function $x(t)$ for when $m = 3$, $c = 30$, $k = 63$, $x_0 = 2$ and $\nu_0 = 2$. Is the motion overdamped, critically damped, or underdamped? If underdamped, write the position function in the form $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$. Also, find the undamped position function $u(t) = C_0\cos(\omega_0 t - \alpha_0)$ if c had been 0. Look at the different graphs for $x(t)$ and $u(t)$ to compare.



Dashpot provides resistance proportional to velocity

$F_R = -cv = -cx'(t)$

resistance

c is positive damping coefficient

Total force is $F_T = m \cdot a = mx''(t)$

The spring force is proportional to the distance of the mass from the Equilibrium position

$F_S = -kx$

so k is a positive-value as the spring coefficient

Let F_E be some external force.

Then: $F_T = F_R + F_S + F_E$

$\Rightarrow \quad mx'' = -cx' - kx + F_E$

$\Rightarrow \quad mx'' + cx' + kx = F_E$

If $F_E = 0$ then:

$mx'' + cx' + kx = 0$

$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$

$r^2 + \frac{c}{m}r + \frac{k}{m} = 0 \quad \Rightarrow r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

No Damping

Case 1: $c = 0 \quad r = \frac{\pm\sqrt{-4km}}{2m} = \pm\sqrt{\frac{k}{m}}\,i$

$\omega_0 = \sqrt{\frac{k}{m}}$

$x(t) = A\cos\left(\sqrt{\tfrac{k}{m}}t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}t\right)$

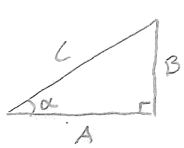
$= A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$= C\left(\frac{A}{C}\cos(\omega_0 t) + \frac{B}{C}\sin(\omega_0 t)\right)$

$= C\left(\cos(\alpha)\cos(\omega_0 t) + \sin(\alpha)\sin(\omega_0 t)\right)$

$= C \cdot \cos(\omega_0 t - \alpha)$

Amplitude, frequency, phase angle/shift



$A^2 + B^2 = C^2$

$\cos(\alpha) = \frac{A}{C}$

$\sin(\alpha) = \frac{B}{C}$

Pay attention to signs of A + B to determine α

OVER

Case 2: $c^2 > 4km$ then $r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$ are real

Over Damping

so $x(t) = c_1 e^{\frac{-c+\sqrt{c^2-4km}}{2m}t} + c_2 e^{\frac{-c-\sqrt{c^2-4km}}{2m}t}$

since $\sqrt{c^2-4km} < c$, both exponents are negative so $x \to 0$ as $t \to \infty$.

Case 3: $c^2 = 4km$ then $r = \frac{-c}{2m}$ so

Critical Damping

$x(t) = c_1 e^{-\frac{c}{2m}t} + c_2 t e^{-\frac{c}{2m}t} = e^{-\frac{c}{2m}t}(c_1 + c_2 t)$

Again $x \to 0$ as $t \to \infty$. Here $c_1 + c_2 t = 0$ has one solution so this passes through equilibrium exactly once.

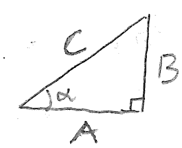
Case 4: $c^2 < 4km$ then $r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$ are imaginary

Under-Damped

$= -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{4km - c^2}\, i$

So $x(t) = e^{-\frac{c}{2m}}(A\cos(\omega_1 t) + B\sin(\omega_1 t))$ with $\omega_1 = \frac{\sqrt{4km - c^2}}{2m}$

$= Ce^{-c/2m} \cdot \cos(\omega_1 t - \alpha)$ where



If $m = 3$, $c = 30$, $k = 63$, $x_0 = 2$, $v_0 = 2$

Then $c^2 - 4km = 900 - 12 \cdot 63 = 144 \Rightarrow r = \frac{-30 \pm 12}{6} = -3, -7$

Overdamped.

$x(t) = Ae^{-7t} + Be^{-3t}$

$2 = A + B$

$v(t) = -7Ae^{-7t} - 3Be^{-3t}$

$2 = -7A - 3B$

$2 = -7(2 - B) - 3B = -14 + 4B$

$B = 4 \qquad A = -2$

$x(t) = -2e^{-7t} + 4e^{-3t}$

If $c = 0$: $r = \pm\sqrt{\frac{k}{m}}\, i = \pm \omega_0 i = \pm\sqrt{21}\, i$

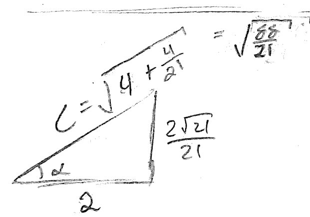
$u(t) = A\cos(\sqrt{21}t) + B\sin(\sqrt{21}t)$

$2 = A \qquad u'(t) = -A\sqrt{21}\sin(\sqrt{21}t) + B\sqrt{21}\cos(\sqrt{21}t)$

$B = \frac{2\sqrt{21}}{21} \qquad 2 = B\sqrt{21}$

$u(t) = 2\cos(\sqrt{21}t) + \frac{2\sqrt{21}}{21}\sin(\sqrt{21}t)$

$u(t) = \sqrt{\frac{88}{21}}\cos\left(\sqrt{21}t - \tan^{-1}\left(\frac{\sqrt{21}}{21}\right)\right)$



$\alpha = \tan^{-1}\left(\frac{\sqrt{21}}{21}\right)$

2. Determine the general solution to the following non-homogeneous differential equations.

a. $y'' + 4y = 3x^3$.

$r^2 + 4 = 0$

$(r - 2i)(r + 2i) = 0$

$r = \pm 2i$

$y_c = C_1\cos(2x) + C_2\sin(2x)$

$y_p = Ax^3 + Bx^2 + Cx + D$

$y_p' = 3Ax^2 + 2Bx + C \qquad y_p'' = 6Ax + 2B$

$y_p'' + 4y = 6Ax + 2B + 4Ax^3 + 4Bx^2 + 4Cx + 4D$

$= 4Ax^3 + 4Bx^2 + (6A + 4C)x + (2B + 4D) = 3x^3$

$4A = 3 \qquad 4B = 0 \qquad 6A + 4C = 0 \qquad 2B + 4D = 0$

$A = 3/4 \qquad B = 0 \qquad 9/2 + 4C = 0 \qquad 4D = 0$

$C = -\frac{9}{8} \qquad D = 0$

$y_p = \frac{3}{4}x^3 - \frac{9}{8}x \qquad y_g = C_1\cos(2x) + C_2\sin(x) + \frac{3}{4}x^3 - \frac{9}{8}x$

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b. $y^{(4)} - 4y'' = x^2$.

$r^4 - 4r^2 = 0$

$r^2(r^2 - 4) = 0$

$r_1 = r_2 = 0 \quad r_3 = 2, \; r_4 = -2$

$y_c = c_1x + c_2 + c_3e^{2x} + c_4e^{-2x}$

$y_{P_{trial}} = Ax^2 + \underbrace{Bx + C}_{\text{Duplicated in } y_c}$

When a portion of $y_{P_{trial}}$ is in y_c, multiply by x^k so that all terms are no longer in y_c:

$y_p = Ax^4 + Bx^3 + Cx^2$

$y_p' = 4Ax^3 + 3Bx^2 + 2Cx \qquad y_p'' = 12Ax^2 + 6Bx + 2C$

$y_p''' = 24Ax + 6B \qquad y_p^{(4)} = 24A$

$y_p^{(4)} - 4y_p'' = 24A - 48Ax^2 - 24Bx - 8C$

$= -48Ax^2 - 24Bx + 24A - 8C = x^2$

$-48A = 1 \qquad -24B = 0 \qquad 24A - 8C = 0$

$A = -\frac{1}{48} \qquad B = 0 \qquad -\frac{1}{2} - 8C = 0$

$C = 1/16$

$y_p(x) = -\frac{1}{48}x^4 + \frac{1}{16}x^2$

$y_g = c_1x + c_2 + c_3e^{2x} + c_4e^{-2x} - \frac{1}{48}x^4 + \frac{1}{16}x^2$

3. Solve the following initial value problem:

$$y^{(3)} + 6y'' + 9y' = \cancel{3x + 1 + e^{-3x} - 2xe^{-3x} + 2x^3e^{-3x}};\ y(0) = 2,\ y'(0) = -2,\ y''(0) = 21$$

$$27 + 12e^{-3x} - 36xe^{-3x}$$

$r^3 + 6r^2 + 9r = 0$

$r(r+3)^2 = 0$

$r_1 = 0 \quad r_2 = r_3 = -3$

$y_c = c_1 + c_2e^{-3x} + c_3xe^{-3x}$

$$y_{trial} = \underbrace{(A)}_{\text{Duplicated}} + \underbrace{\left(Be^{-3x} + Cxe^{-3x}\right)}_{\text{separate Duplication}}$$

$y_p = Ax + Bx^2e^{-3x} + Cx^3e^{-3x}$

$y_p' = A + 2Bxe^{-3x} + (-3B + 3C)x^2e^{-3x} - 3Cx^3e^{-3x}$

$y_p'' = 2Be^{-3x} + (-12B + 6C)xe^{-3x} + (9B - 18C)x^2e^{-3x} + 9Cx^3e^{-3x}$

$y_p''' = (-18B + 6C)e^{-3x} + (54B - 36C)xe^{-3x} + (-27B + 81C)x^2e^{-3x}$
$-27Cx^3e^{-3x}$

$$\left.\begin{aligned} y_p^{(3)} + 6y_p'' + 9y_p' &= 9A + (-6B + 6C)e^{-3} - 18Cxe^{-3x} \\ &= 27 + 12e^{-3x} - 36xe^{-3x} \end{aligned}\right\} \Rightarrow \quad \begin{array}{lll} 9A = 27 & -6B + 6C = 12 & -18C = -36 \\ A = 3 & -6B = 0 & C = 2 \\ & B = 0 & \end{array}$$

$\Rightarrow \quad y_p = 3x + 2x^3e^{-3x}$

$y_g = c_1 + c_2e^{-3x} + c_3xe^{-3x} + 3x + 2x^3e^{-3x}$

$y_g' = (-3c_2 + c_3)e^{-3x} + (-3c_3)xe^{-3x} + 6x^2e^{-3x} - 6x^3e^{-3x} + 3$

$y_g'' = (9c_2 - 6c_3)e^{-3x} + (9c_3 + 6)xe^{-3x} - 36x^2e^{-3x} + 18x^3e^{-3x}$

$y(0) = 2 \Rightarrow c_1 + c_2 = 2$

$y'(0) = -2 \Rightarrow -3c_2 + c_3 + 3 = -2$

$-3c_2 + c_3 = -5$

$y''(0) = 21 \Rightarrow 9c_2 - 6c_3 = 21$

$3c_2 - 2c_3 = 7$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -3 & 1 & -5 \\ 0 & 3 & -2 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$\Rightarrow \quad c_1 = 1 \quad c_2 = 1 \quad c_3 = -2$

$y(x) = 1 + e^{-3x} - 2xe^{-3x} + 3x + 2x^3e^{-3x}$

$= 1 + 3x + e^{-3x} - 2xe^{-3x} + 2x^3e^{-3x}$