Math 253 LP 7 - 8.6 Representations of Functions as Power Series

1. Find a power series representation for the function $f(x) = \dfrac{1}{(1-x)^2}$ and determine the radius and interval of convergence.

Note $P(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
$= 1 + x + x^2 + \cdots$
with $I_P = (-1,1)$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = \frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}\frac{1}{1-x}$$

$I_P = (-1,1)$

$$= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty}(n+1)x^n$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

Better

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

Power Rule

$$= \sum_{n=0}^{\infty} n x^{n-1}$$

issue with 1st term

What's the connection?

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(n+1)x^n}{n x^{n-1}}\right| = |x| < 1 \Rightarrow R = 1$$

$x = -1$: $\sum_{n=1}^{\infty} n(-1)^{n-1}$ Divergent (Div. Test)

$x = 1$: $\sum_{n=1}^{\infty} n$ Divergent (Div. Test)

$I_f = (-1,1)$

2. Find a power series representation for the function $f(x) = \tan^{-1}(x)$ and determine its radius and interval of convergence.

$$f(x) = \tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx \qquad \frac{1}{1+x^2} = P(-x^2) = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$= \int 1 - x^2 + x^4 - x^6 + \cdots \, dx$$

$$= \left(x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots\right) + C$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1} x^{2n+1}$$

[Note $C = 0$ since $\tan^{-1}(0) = 0$]

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\cdot\frac{2n+1}{x^{2n+1}}\right| = |x^2| < 1 \Rightarrow R = 1$$

$x = -1$: $\sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{1}{2n+1}$ converges (AST)

$x = 1$: $\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{2n+1}$ " "

$I_f = [-1,1]$

3. Find a power series representation for the function $f(x) = \dfrac{x}{2x^2+1}$ and determine its radius and interval of convergence.

$$f(x) = \frac{x}{2x^2+1} = x\cdot\frac{1}{1-(-2x^2)} = x\cdot P(-2x^2) \xrightarrow{OR} = x\left(1 + (-2x^2) + (-2x^2)^2 + (-2x^2)^3 + (-2x^2)^4 + \cdots\right)$$

$$= x\cdot\sum_{n=0}^{\infty}(-2x^2)^n \qquad = x\left(1 - 2x^2 + 2^2x^4 - 2^3x^6 + 2^4x^8 - \cdots\right)$$

$$= x\cdot\sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot x^{2n} \qquad = x - 2x^3 + 2^2x^5 - 2^3x^7 + 2x^9 - \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot x^{2n+1} \qquad = \sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot x^{2n+1}$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{2^{n+1}\cdot x^{2n+3}}{2^n\cdot x^{2n+1}}\right| = |2x^2| < 1$$

$\Rightarrow$ $|x^2| < \frac{1}{2}$

$|x| < \frac{\sqrt{2}}{2} = R$

$-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$

$x = -\frac{\sqrt{2}}{2}$: $\sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot\left(-\frac{\sqrt{2}}{2}\right)^{2n+1} = \sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot\left(\frac{1}{2}\right)^n\cdot\frac{-\sqrt{2}}{2} = \sum_{n=0}^{\infty}\left(-\frac{1}{\sqrt{2}}\right)(-1)^n$ Divergent (Div. Test)

$x = \frac{\sqrt{2}}{2}$: $\sum_{n=0}^{\infty}(-1)^n\cdot 2^n\left(\frac{\sqrt{2}}{2}\right)^{2n+1} = \sum_{n=0}^{\infty}(-1)^n\cdot 2^n\cdot\left(\frac{1}{2}\right)^n\cdot\frac{\sqrt{2}}{2} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{\sqrt{2}}{2}$ " "

$I = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

4. Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int \frac{t}{1-t^8}dt = \int t \cdot P(t^8)\,dt$$
$$= \int t \sum_{n=0}^{\infty} (t^8)^n\,dt$$
$$= \int \sum_{n=0}^{\infty} t^{8n+1}\,dt$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{8n+2} t^{8n+2}\right) + C$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{t^{8n+10}}{8n+10} \cdot \frac{8n+2}{t^{8n+2}} \right| = |t^8| < 1 \Rightarrow |t| < 1 = R$$

$t=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^{8n+2}}{8n+2} = \sum_{n=0}^{\infty} \frac{1}{8n+2}$ Diverges $(p=1)$

$t=1$: $\sum_{n=0}^{\infty} \frac{1}{8n+2}$ Diverges $(p=1)$

$I = (-1, 1)$

5. Use a power series to approximate the definite integral to three decimal places.

$$\int_0^{0.3} x\arctan(3x)dx = \int_0^{0.3} \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} \cdot x^{2n+2}}{2n+1}\,dx$$

Note: $\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$\Rightarrow x\tan^{-1}(3x) = x \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} \cdot x^{2n+2}}{2n+1}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} \cdot x^{2n+3}}{(2n+3)(2n+1)} \Bigg|_0^{0.3}$$

$$= \left(\frac{3x^3}{3\cdot 1} - \frac{3^3x^5}{5\cdot 3} + \frac{3^5x^7}{7\cdot 5} - \frac{3^7x^9}{9\cdot 7} + \cdots\right)\Bigg|_0^{0.3}$$

$$= \underbrace{(0.3)^3 - \frac{3^2}{5}(0.3)^5 + \frac{3^5}{7\cdot 5}(0.3)^7 - \underbrace{\frac{3^7}{9\cdot 7}(0.3)^9}_{\approx 0.0007} + \frac{3^9}{11\cdot 9}(0.3)^{11}}_{} - \frac{3^{11}}{13\cdot 11}(0.3)^{13} + \underbrace{\frac{3^{13}}{15\cdot 13}(0.3)^{15}}_{0.00011} - \cdots$$

≈ 0.624

6. Starting with the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$ and then use it to find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{6n+3}(2n+1)}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan(x)$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} = \arctan(x^3)$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{6n+3}(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})^{6n+3}}{(2n+1)} = \arctan((\tfrac{1}{2})^3) = \arctan(\tfrac{1}{8})$$