Math 253 LP 8 - 8.7 Introduction to Taylor and Maclaurin Series   Solutions

1. Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. What is the radius of convergence for each?

a. $f(x) = e^x$ $f(0)=1$

$f'(x)=e^x$ $f'(0)=1$

$f''(x)=e^x$ $f''(0)=1$

$\vdots$ $\vdots$

$f^{(n)}(x)=e^x$ $f^{(n)}(0)=1$

By Maclaurin: $M(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$= \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \cdots$

Thus $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Note $0! = 1$

For radius of conv.: $\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$ so converges on $(-\infty,\infty)$ $(R=\infty)$

b. $f(x) = \sin(x)$ $f(0)=0$

$f'(x)=\cos(x)$ $f'(0)=1$

$f''(x)=-\sin(x)$ $f''(0)=0$

$f'''(x)=-\cos(x)$ $f'''(0)=-1$

$\vdots$

Thus

$\sin(x) = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$

$= \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$

$\lim_{k\to\infty}\left|\frac{x^{2(k+1)+1}}{(2(k+1)+1)!}\cdot\frac{(2k+1)!}{x^{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{(2k+3)(2k+2)}\right| = 0$ so $R=\infty$

2. Find the Taylor series for $f(x)$ centered at the given value of a.

a. $f(x) = \cos(x), \quad a = \pi$

By Taylor: $T_a(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

That is: $T_a(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$f(\pi) = -1$

$f'(x) = -\sin(x) \quad f'(\pi) = 0$

$f''(x) = -\cos(x) \quad f''(\pi) = 1$

$f'''(x) = \sin(x) \quad f'''(\pi) = 0$

$\vdots$

Thus

$$\cos_\pi(x) = \frac{-1}{0!} + \frac{0}{1!}(x-\pi) + \frac{1}{2!}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3 + \frac{-1}{4!}(x-\pi)^4 + \cdots$$

$$= -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k)!}(x-\pi)^{2k}$$

[Again $R = \infty$]

b. Explain why the formulas for Maclaurin and Taylor series work.

~~If~~ If there exists a power series for a function, $f(x)$ it is of the form $T(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

(a power series shifted to center at "a")

Now $T'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$ so $T'(a) = c_1$

$T''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots$ so $T''(a) = 2c_2$

$T'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots$ so $T'''(a) = 3\cdot 2c_3$

Continuing we get $T^{(n)}(a) = n!\,c_n$

Since we want $T(x) = f(x)$, we also want $T^{(n)}(x) = f^{(n)}(x)$ for all n.

Thus $f^{(n)}(a) = T^{(n)}(a) = n!\,c_n$ so $c_n = \frac{f^{(n)}(a)}{n!}$

& $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

3. Prove that the Maclaurin series, $M(x)$, for $\sin(x)$ converges to $\sin(x)$. That is, prove that $M(x) = f(x)$ for all x.

$$M(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

By Taylor's Inequality:

If $|f^{(n+1)}(x)| \leq K$ for $|x-a| \leq d$,

then $|R_n(x)| \leq \frac{K}{(n+1)!} |x-a|^{n+1}$ on $|x-a| \leq d$

Here $f(x) = \sin(x)$ & $|f^{(n+1)}(x)| \leq 1$ for all n & all $x \in \mathbb{R}$

Thus $|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1}$

Now $\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$

$= 0$

Note $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x by ratio-test (see # 1)

thus $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$ converges

& $\frac{|x|^n}{n!} \to 0$ By divergence test.

Thus, if the Taylor series has an infinite number of terms, it has a remainder of 0 when compared to $\sin(x)$. Thus $M(x) = \sin(x)$