Solutions

Math 253 LP ~~XX~~ 9 - 8.7 More on Representations of Functions as Power Series

You need to have the first 4 of these memorized.

- $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ $\qquad R = 1$

- $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ $\qquad R = \infty$

- $\sin(x) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$ $\qquad R = \infty$

- $\cos(x) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$ $\qquad R = \infty$

- $\tan^{-1}(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$ $\qquad R = 1$

- $\ln(1+x) = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$ $\qquad R = 1$

- $(1+x)^k = \sum\limits_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \ldots$ $\qquad R = 1$

1. Use a Maclaurin series from the table above to obtain the Maclaurin series for the function $f(x) = \dfrac{x^2}{\sqrt{2+x}}$.

$$f(x) = x^2 \cdot \left(2\left(1+\tfrac{x}{2}\right)\right)^{-1/2} = x^2 \cdot 2^{-1/2}\left(1+x/2\right)^{-1/2}$$

$$= x^2/\sqrt{2}\left(1 + \left(-\tfrac{1}{2}\right)\left(\tfrac{x}{2}\right) + \frac{(-1/2)(-3/2)}{2!}\left(\tfrac{x}{2}\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(\tfrac{x}{2}\right)^3 + \cdots\right)$$

$$= \frac{x^2}{\sqrt{2}}\left(1 - \frac{1}{2^2}x + \frac{3}{2!\cdot 2^4}x^2 - \frac{3\cdot 5}{3!\cdot 2^6}x^4 + \cdots\right) = \frac{\sqrt{2}}{2}\left(x^2 - \frac{1}{2^2}x^3 + \frac{3}{2!\cdot 2^4}x^4 - \frac{3\cdot 5}{3!\cdot 2^6}x^5 + \cdots\right)$$

0 1 2 3

$$= \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(1\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1))}{n!\cdot 2^{2n}}\cdot x^{n+2}$$

2. Use series to approximate the definite integral to within five decimal places.

$\int_0^{0.2} [\arctan(x^3) + \sin(x^3)]dx$

$$= \int_0^{0.2} \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} dx$$

$$= \int_0^{0.2} \left(\frac{1}{1}x^3 - \frac{1}{3}x^9 + \frac{1}{5}x^{15} - \cdots\right) + \left(\frac{1}{1!}x^3 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{15} - \cdots\right) dx$$

$$= \int_0^{0.2} (1+1)x^3 - \left(\frac{1}{3} + \frac{1}{3!}\right)x^9 + \left(\frac{1}{5} + \frac{1}{5!}\right)x^{15} - \cdots)$$

$$= \frac{(1+1)x^4}{4} - \frac{(1/3 + 1/3!)}{10}x^{10} + \frac{(1/5 + 1/5!)}{16}x^{16} - \cdots \Big]_0^{0.2} = \frac{2}{4}(0.2)^4 - \frac{(1/3 + 1/3!)}{10}(0.2)^{10} + \frac{(1/5 + 1/5!)}{16}(0.2)^{16} - \cdots$$

0.0000000000512

≈ 0.00080

3. Use division of power series to find the first three nonzero terms in the Maclaurin series for the function $y = \sec(x)$.

$$\sec(x) = \frac{1}{\cos(x)} = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} = 1 + \frac{1}{2!}x^2 + \left(\frac{1}{4!} + \frac{1}{2!\cdot 2!}\right)x^4 + \cdots$$

$$\begin{array}{r|l} & 1 + \frac{1}{2!}x^2 + \left(\frac{1}{4!} + \frac{1}{2!2!}\right)x^4 + \cdots \\ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots & 1 + 0x^2 + 0x^4 + 0x^6 + \cdots \\ & -\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) \\ \hline & \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots \\ & -\left(\frac{1}{2!}x^2 - \frac{1}{2!\cdot 2!}x^4 + \frac{1}{2!\cdot 4!}x^6 - \cdots\right) \\ \hline & \left(\frac{1}{4!} + \frac{1}{2!2!}\right)x^4 + \left(\frac{1}{6!} - \frac{1}{2!\cdot 4!}\right)x^6 + \cdots \end{array}$$

4. Find the sum for the series $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n \cdot n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}$$