
LESSON 1

Vector Valued Functions

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1.1 Introduction to Vector Valued Functions

From previous classes, the topics of vectors, planes, and functions have been covered. We may now link these ideas together by defining a function in 3D-space in terms of some parameter t , for example. This is merely a natural progression from parameterization of a function in 2D-space.

Definition 1.1.1

A **Vector-Valued Function** is a function with a single-variable input and a vector output. That is, it is a function that outputs a vector whose components are determined by the same input variable.

Note: Vector-Valued Functions are more simply just parametric equations but written in vector notation.

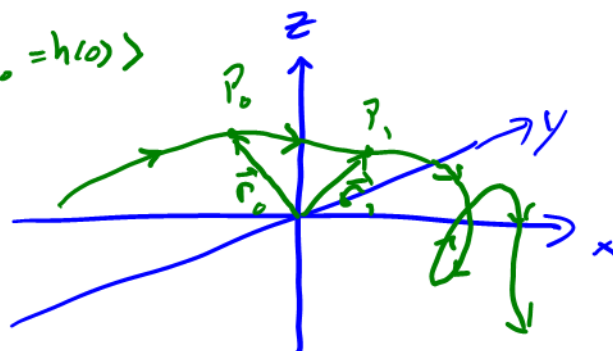
Example 1.1.1 Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ where f, g , and h are real-valued functions. Function $\mathbf{r}(t)$ is a **vector function**.

$$\vec{r}(0) = \vec{r}_0 = \langle x_0 = f(0), y_0 = g(0), z_0 = h(0) \rangle$$

$$\sim P_0 = (x_0, y_0, z_0)$$

$$\vec{r}(1) = \vec{r}_1 = \langle f(1), g(1), h(1) \rangle$$

$$\sim P_1 = (x_1, y_1, z_1)$$



Notes about the function and curve:

- The function defines a curve in 3D-space
- This type of function arises in many physical applications, for example in fluid mechanics.
- The arrows on the curve represent the direction for which the curve is generated as t increases.

Example 1.1.2 Determine the domain of the vector-valued function

$$\mathbf{r}(t) = \langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \rangle$$

$$D = \{t \in \mathbb{R} \mid -1 < t \leq 2\}$$

$$= (-1, 2]$$

$$\begin{aligned} 4-t^2 &\geq 0 \quad \text{AND} \quad t+1 > 0 \\ -t^2 &\geq -4 && t > -1 \\ t^2 &\leq 4 \\ -2 &\leq t \leq 2 \end{aligned} \quad \swarrow \searrow \quad -1 < t \leq 2$$

Exercise 1.1.1 Determine the domain of the vector-valued function

$$\mathbf{r}(t) = \left\langle \sqrt{t-5}, 3t^2+5, \frac{1}{t} \right\rangle$$

$$t-5 \geq 0 \quad \text{AND} \quad t \neq 0$$

$$t \geq 5 \quad \text{AND} \quad t \neq 0$$


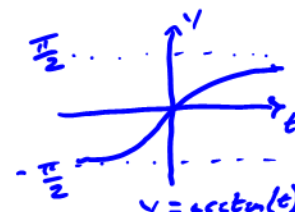
$$\Rightarrow D = \{t \mid t \geq 5\} = [5, \infty)$$

Example 1.1.3 Find $\lim_{t \rightarrow \infty} \langle 2+t^2, \ln(\frac{1}{t}), e^{\arctan(t)} \rangle$.

$$\lim_{t \rightarrow \infty} \langle 2+t^2, \ln(\frac{1}{t}), e^{\arctan(t)} \rangle = \left\langle \lim_{t \rightarrow \infty} (2+t^2), \lim_{t \rightarrow \infty} \ln(\frac{1}{t}), \lim_{t \rightarrow \infty} e^{\arctan(t)} \right\rangle$$

$$= \langle \infty, -\infty, e^{\pi/2} \rangle$$

$\lim_{t \rightarrow \infty} (2+t^2) = \infty$
 $\lim_{t \rightarrow \infty} \ln(\frac{1}{t}) = \lim_{A \rightarrow 0^+} \ln(A) = -\infty$
 let $A = \frac{1}{t}$
 $\lim_{t \rightarrow \infty} e^{\arctan(t)} = e^{\pi/2}$

Exercise 1.1.2 Find $\lim_{t \rightarrow \infty} \left\langle \frac{1+t^2}{1-t^2}, \tan^{-1}(t), \frac{1-e^{-2t}}{t} \right\rangle$.

$$\lim_{t \rightarrow \infty} \frac{1+t^2}{1-t^2} = \lim_{t \rightarrow \infty} \frac{t^2}{-t^2} = -1$$

$$\lim_{t \rightarrow \infty} \frac{1-e^{-2t}}{t}$$

$$= \lim_{t \rightarrow \infty} \frac{1 - \frac{1}{e^{2t}}}{t} \quad \text{Scratch}$$

$$\frac{1-0}{\infty} = 0$$

$$= 0$$

$$\lim_{t \rightarrow \infty} \left\langle \frac{1+t^2}{1-t^2}, \tan^{-1}(t), \frac{1-e^{-2t}}{t} \right\rangle$$

$$= \langle -1, \pi/2, 0 \rangle$$

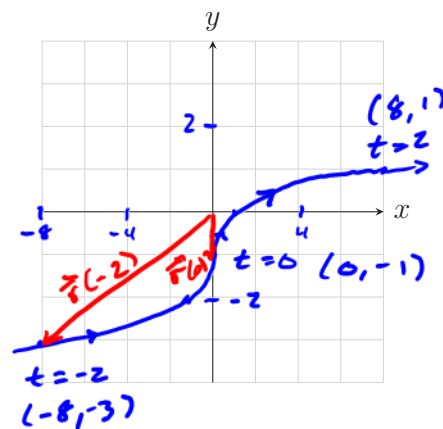
1.2 Graphing Vector Valued Functions

Example 1.2.1 Sketch the curve given by the vector function $\mathbf{r}(t) = \langle t^3, t - 1 \rangle$.

t	$x = t^3$	$y = t - 1$
-2	-8	-3
-1	-1	-2
0	0	-1
1	1	0
2	8	1

$$x = t^3$$

$$y = t - 1$$



Exercise 1.2.1 Sketch the curve given by the vector function $\mathbf{r}(t) = \langle \cos(t), t^2 \rangle$.

t	$x = \cos(t)$	$y = t^2$
0	1	0
$\pm \pi/2$	0	$\frac{1}{4}\pi^2$
$\pm \pi$	-1	π^2
$\pm 3\pi/2$	0	$\frac{9}{4}\pi^2$
$\pm 2\pi$	1	$4\pi^2$

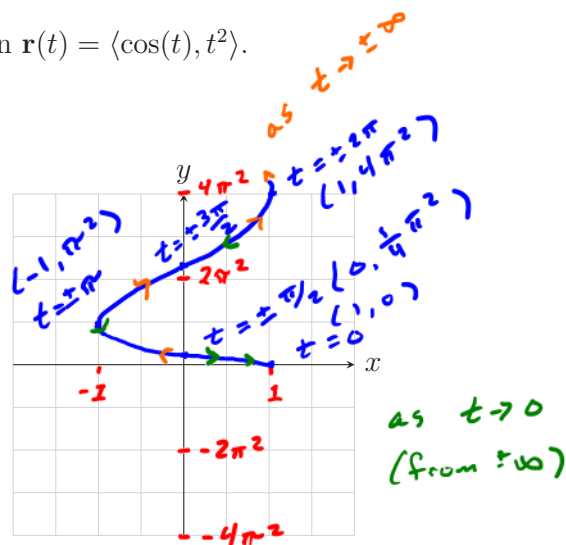
$$(1, 0)$$

$$(0, \frac{1}{4}\pi^2)$$

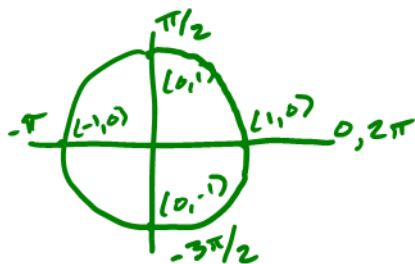
$$(-1, \pi^2)$$

$$(0, \frac{9}{4}\pi^2)$$

$$(1, 4\pi^2)$$



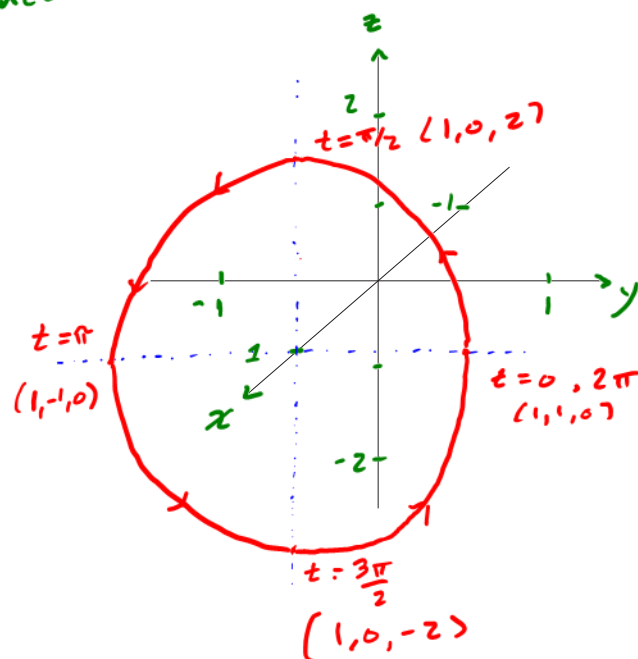
as $t \rightarrow 0$
(from $+\infty$)



Example 1.2.2 Sketch the curve give by the vector function $\mathbf{r}(t) = \langle 1, \cos(t), 2 \sin(t) \rangle$.

$y = \cos(t)$ $z = 2 \sin(t)$

t	x	y	z
0	1	1	0
$\pi/2$	1	0	2
π	1	-1	0
$3\pi/2$	1	0	-2
2π	1	1	0



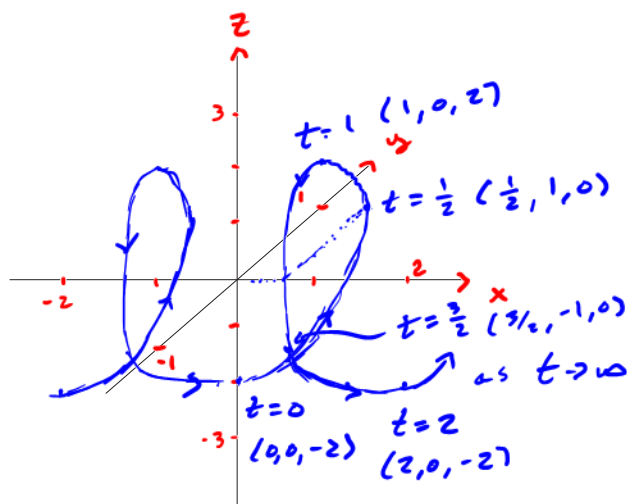
Confirm With Geogebra <https://www.geogebra.org/3d/cbaex6tq>

Exercise 1.2.2 Sketch the curve give by the vector function $\mathbf{r}(t) = t\mathbf{i} + \sin(\pi t)\mathbf{j} - 2 \cos(\pi t)\mathbf{k}$.

$T = \frac{2\pi}{\pi} = 2$
 $T = \frac{2\pi}{B}$ is the period
 $\sin(Bt)$

$x = t$ $y = \sin(\pi t)$ $z = -2 \cos(\pi t)$

t	x	y	z
0	0	0	-2
$1/2$	$1/2$	1	0
1	1	0	2
$3/2$	$3/2$	-1	0
2	2	0	-2



Confirm With Geogebra <https://www.geogebra.org/3d/zand8x4x>

For the following example and exercise, we are going to utilize the Geogebra software to create 3-dimensional graphs of space curves. To access Geogebra3D you can either google Geogebra and then click on the 3D Graphing, or you may go to [geogebra.org](https://www.geogebra.org) and then select 3D Graphing under “New Math Apps.” Another option is to download the Geogebra 3D app to your device. (Caution: confirm file size and space prior to installation.) The instructions for creating 3D Space Curves in Geogebra are explained in the following example.

Example 1.2.3 Use Software to view the following vector-valued function in \mathbb{R}^3 .

$$\mathbf{r}(t) = \langle \sin(2t), e^t, \cos(2t) \rangle$$

Instructions:

- Access Geogebra 3D as explained above.
- In the Input box on the left, enter $(\sin(2t), e^t, \cos(2t))$ and then hit enter.

Geogebra will automatically convert your entry into a “Curve” object and give it a name. In this case, as this is our first input, it will name it “a,” which we can use to call on this vector-valued function. That is, if we now enter into the next Input line “a(3),” Geogebra will automatically output the point when t is 3 on our space curve. Note also that Geogebra will automatically set a domain for the t -values which we may edit by clicking on the greyed-out input box ‘Input...’ and adjusting the start and end values of the domain. The syntax for the Curve command in Geogebra is

$$a = \text{Curve}((x(t), y(t), z(t)), t, \text{start}, \text{end}).$$

Next we want to be able to visibly see the direction of motion of our space curve so we will, in the next Input box, do the following:

- Enter “a(k)” into the Input box and then click “create slider.”

Geogebra will automatically create the point “A” on the curve and a slider for “k” and you may then drag the slider back and forth to see the direction of motion of our curve.

You may use the following link to access the completed graph:

Confirm graph using Geogebra <https://www.geogebra.org/3d/s6pytdpw>

Exercise 1.2.3 Use Software to view the following vector-valued function in \mathbb{R}^3 .

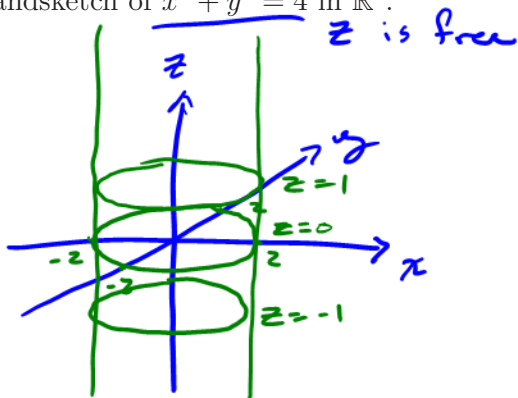
$$\mathbf{r}(t) = \langle t^2, \cos(4t), \sin(3t) \rangle$$

Confirm graph using Geogebra <https://www.geogebra.org/3d/whyu8vna>

1.3 Intersections

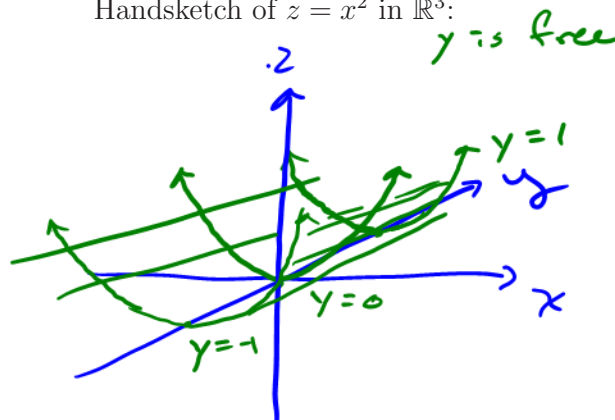
Example 1.3.1 Consider the two surfaces $x^2 + y^2 = 4$ and $z = x^2$. Analytically determine the parametric equations for the curve of intersection of the two surfaces.

Handsketch of $x^2 + y^2 = 4$ in \mathbb{R}^3 :

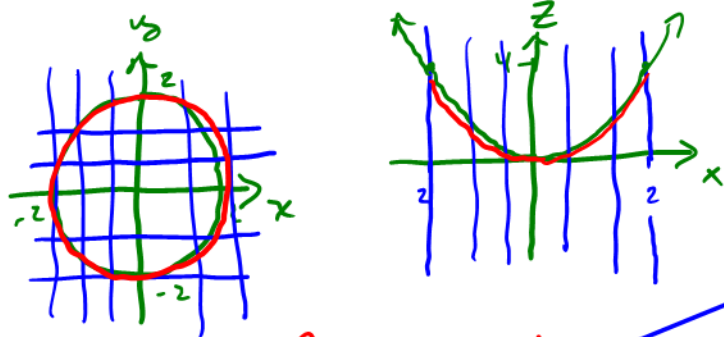


$x = \cos(t)$
 $y = \sin(t)$
 ⊕ unit circle

Handsketch of $z = x^2$ in \mathbb{R}^3 :



Handsketch of both curves from above, looking down on the xy -plane.



red: circle of intersection
 in x or y
 $x = 2\cos(t)$ $y = 2\sin(t)$

So $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos^2(t) \rangle$

Substitute the two surfaces into each other to determine the vector-valued function for the space curve, $\mathbf{r}(t)$.

$$\begin{aligned}
 z &= x^2 \\
 z &= (2\cos(t))^2 \\
 &= 4\cos^2(t)
 \end{aligned}$$

When looking for intersections, we will need to use substitution in some manner.

Confirm Graph Using Geogebra <https://www.geogebra.org/3d/j73c9ecj>

Example 1.3.2 At what point(s) does the curve $\mathbf{r}(t) = \langle \sin(t), \cos(t), t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

$$\sin^2(t) + \cos^2(t) + t^2 = 5$$

$$1 + t^2 = 5$$

$$t^2 = 4$$

$$t = \pm 2$$

$x \quad y \quad z$
circles in $x \text{ or } y$

$$\vec{r}(-2) = \langle \sin(-2), \cos(-2), -2 \rangle$$

$$\vec{r}(2) = \langle \sin(2), \cos(2), 2 \rangle$$



Confirm Graph Using Geogebra <https://www.geogebra.org/3d/rghvgdjy>

Example 1.3.3 Given the two space curves

$$\mathbf{r}(t) = \langle 2t + 1, -t + 2, t + 3 \rangle \text{ and } \mathbf{q}(s) = \langle -3s + 8, s - 1, -2s + 7 \rangle,$$

determine if they intersect and, if so, determine if it is a collision.

For two lines to intersect, the $x, y, \text{ or } z$ -values all need to be equal.

$$2t + 1 = -3s + 8$$

$$\text{If } s = 1 \text{ or } t = 2:$$

$$2(2) + 1 \stackrel{?}{=} -3(1) + 8$$

$$5 = 5 \quad \checkmark$$

$$-t + 2 = s - 1$$

$$s = -t + 3$$

$$s = -2 + 3$$

$$s = 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$= 1$$

$$t + 3 = -2s + 7$$

$$t + 3 = -2(-t + 3) + 7$$

$$t + 3 = 2t + 1$$

$$2 = t$$

$$\vec{q}(1) = \langle 5, 0, 5 \rangle$$

$$\vec{r}(2) = \langle 5, 0, 5 \rangle$$

Yes, they intersect at the point $(5, 0, 5)$.

No, it is not a collision, since $s \neq t$.

Confirm Graph Using Geogebra <https://www.geogebra.org/3d/dj4gdfda>