LESSON 2

Derivatives and Integrals of Vector Functions

Contents

In Differential Calculus and Integral Calculus, we explored the ideas of differentiation and integration for functions of a single variable that output real numbers. We now extend these concepts to that of vector-valued functions. While the derivative of a vector-valued function has a great correspondence to the derivative of a real-valued function - that is, it gives the direction of the tangent line - the integral of of a vector-valued function is a little bit more abstract as it will produce another vector-valued function, not an area under a curve per se. We will explore some of the applications of this in later sections.

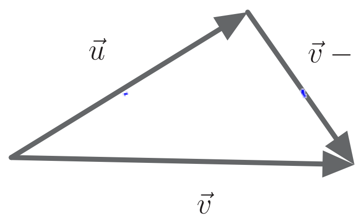
2.1 Tangent Vectors

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

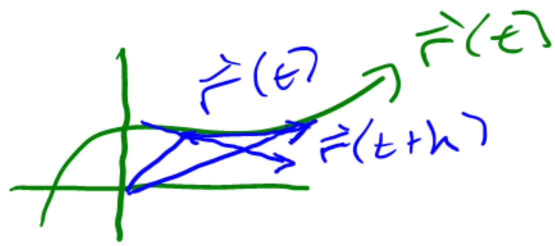
$\sim h$

2.1.1 Background Review

Recall the basic vector operation of subtraction.



Now let $\mathbf{r}(t)$ be a vector function.



In the analogous step from differentiation in one-variable, we define $\mathbf{r}'(t) = \lim_{h \to 0} \dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$.

The procedure for differentiating $\mathbf{r}(t)$ occurs in the natural way — by component.

$$\Delta r = \langle \Delta x, \Delta y, \Delta z \rangle$$

$$\frac{\Delta r}{\Delta t} = \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \right\rangle$$

2.1.2 Derivatives

Example 2.1.1 Find the derivative of the vector function $\mathbf{r}(t) = \langle \cos(3t), t, \sin(3t) \rangle$

$$\vec{r}'(t) = \frac{d}{dt} \langle \cos(3t), t, \sin(3t) \rangle$$

$$= \left\langle \frac{d}{dt} \cos(3t), \frac{d}{dt} t, \frac{d}{dt} \sin(3t) \right\rangle$$

$$= \langle -\sin(3t) \cdot 3, 1, \cos(3t) \cdot 3 \rangle$$

$$= \langle -3\sin(3t), 1, 3\cos(3t) \rangle$$

Definition 2.1.1

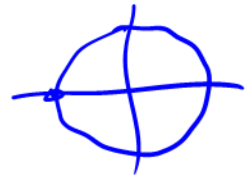
The **Tangent Vector** to a vector-valued function, $\mathbf{r}(t)$ at a given point P, with P on $\mathbf{r}$ at $t = a$, is the vector $\mathbf{r}'(a)$ with initial point at $\mathbf{r}(a)$. The tangent vector at P exists if $\mathbf{r}'(a)$ exists and $\mathbf{r}'(a) \neq \mathbf{0}$.

Example 2.1.2 Find the tangent vector to $\mathbf{r}(t) = \langle \cos(3t), t, \sin(3t) \rangle$ at the point when $t = \pi$

$\vec{r}'(t) = \langle -3\sin(3t), 1, 3\cos(3t) \rangle$

$\vec{r}'(\pi) = \langle -3\sin(3\pi), 1, 3\cos(3\pi) \rangle$

$= \langle 0, 1, -3 \rangle$



2.2 Tangent Lines

$T_{f(a)}(x) = f'(a)(x-a) + f(a)$

Example 2.2.1 Given the vector-valued function $\mathbf{r}(t) = \langle t-2, t^2+1 \rangle$, sketch the plane-curve, find $\mathbf{r}(t)$, sketch the position vectors for $\mathbf{r}(-2)$, $\mathbf{r}(-1)$, and $\mathbf{r}(0)$, sketch the tangent vector $\mathbf{r}'(-1)$, and determine the equation of the tangent line at this location.

t	$\vec{r}(t)$
-2	$\langle -4, 5 \rangle$
-1	$\langle -3, 2 \rangle$
0	$\langle -2, 1 \rangle$
1	$\langle -1, 2 \rangle$
2	$\langle 0, 5 \rangle$

$\vec{r}'(t) = \langle 1, 2t \rangle$

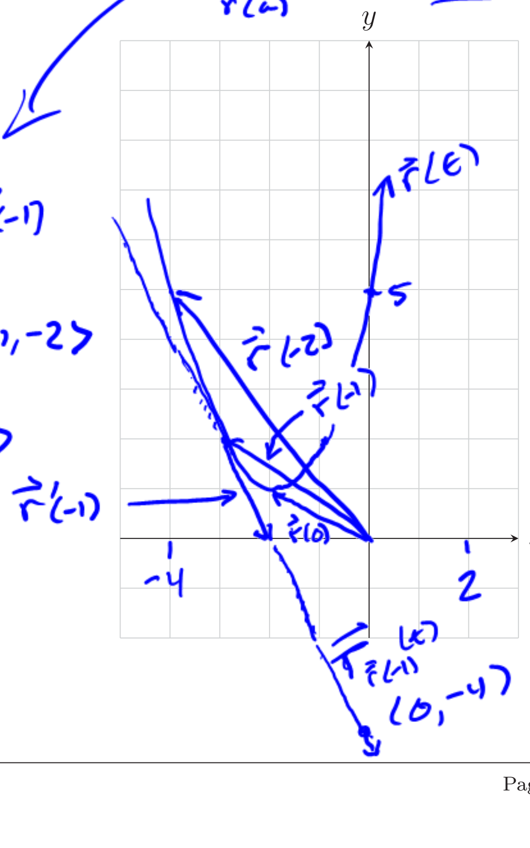
$\vec{r}'(-1) = \langle 1, -2 \rangle$

$T_{\vec{r}(a)}(t) = \vec{r}(a) + t\,\vec{r}'(a)$

$\vec{T}_{\vec{r}(-1)}(t) = \vec{r}(-1) + t\,\vec{r}'(-1)$

$= \langle -3, 2 \rangle + t\langle 1, -2 \rangle$

$= \langle -3+t, 2-2t \rangle$



$y = -2x - 4$

Exercise 2.2.1 Given the vector-valued function $\mathbf{r}(t) = \langle t+1, \sqrt{t}\rangle$, sketch the plane-curve, find $\mathbf{r}(t)$, sketch the position vectors for $\mathbf{r}(1)$, $\mathbf{r}(2)$, and $\mathbf{r}(3)$, sketch the tangent vector $\mathbf{r}'(2)$, and determine the equation of the tangent line at this location.

t	$\vec{r}(t)$
0	$\langle 1,0\rangle$
1	$\langle 2,1\rangle$
2	$\langle 3,\sqrt{2}\rangle$
3	$\langle 4,\sqrt{3}\rangle$
4	$\langle 5,2\rangle$
9	$\langle 10,3\rangle$

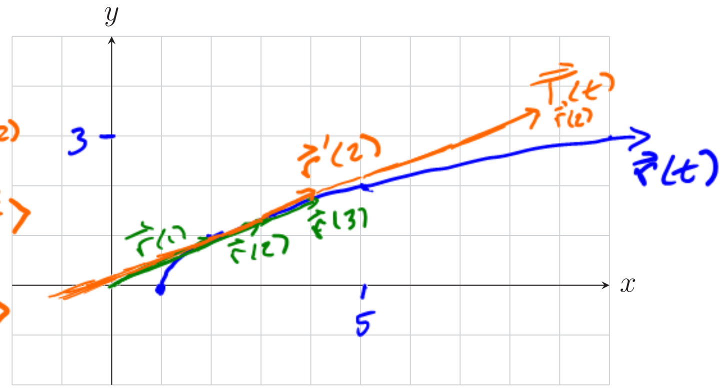
$\vec{r}\,'(t) = \langle 1, \frac{1}{2\sqrt{t}}\rangle$

$\vec{r}\,'(2) = \langle 1, \frac{\sqrt{2}}{4}\rangle$

$\vec{T}_{\vec{r}(2)}(t) = \vec{r}(2) + t\vec{r}\,'(2)$

$= \langle 2,1\rangle + t\langle 1, \frac{\sqrt{2}}{4}\rangle$

$= \langle 2+t, 1+\frac{\sqrt{2}}{4}t\rangle$



Example 2.2.2 Given the vector-valued function $\mathbf{r}(t) = \langle 4\sqrt{t}, t, t^2\rangle$ find $\mathbf{r}'(t)$ then determine the equation of the tangent line at the point $(4, 1, 1)$. Use technology to graph the space curve and tangent line at this point.

$\vec{r}\,'(t) = \langle 2/\sqrt{t}, 1, 2t\rangle$

$\langle 4,1,1\rangle = \langle 4\sqrt{t}, t, t^2\rangle$

$4 = 4\sqrt{t}$

$1 = \sqrt{t}$

$t = 1$

$\vec{r}\,'(1) = \langle 2,1,2\rangle$

$\vec{T}_{\vec{r}(1)}(t) = \vec{r}(1) + t\cdot\vec{r}\,'(1)$

$= \langle 4,1,1\rangle + t\langle 2,1,2\rangle$

$= \langle 4+2t, 1+t, 1+2t\rangle$

To plot these using Geogebra, we will enter

(4sqrt(t),t,t∧2)

to produce the space curve. We then plot the point that we want to produce a tangent line from by entering $(4, 1, 1)$ as the second input. We can enter

Vector((4, 1, 1), (6, 2, 3))

to create the tangent vector from the given point. Finally, we enter

$$(4, 1, 1) + (2, 1, 2)t$$

to plot the entire tangent line.

Confirm Graph Using Geogebra https://www.geogebra.org/3d/kgrgzvsa

Exercise 2.2.2 Given the vector-valued function $\mathbf{r}(t) = \langle t\sin(t), t^2, t\cos(2t)\rangle$ find $\mathbf{r}'(t)$ then determine the equation of the tangent line at the point $(0, 0, 0)$. Use technology to graph the space curve and tangent line at this point.

At $(0,0,0)$, we have $t=0$ (by observation).

$\vec{r}\,'(t) = \langle \sin(t) + t\cos(t), 2t, \cos(2t) - 2t\sin(2t)\rangle$

$\vec{r}\,'(0) = \langle 0, 0, 1\rangle$

$\vec{T}_{\vec{r}(0)}(t) = \langle 0,0,0\rangle + t\langle 0,0,1\rangle = \langle 0, 0, t\rangle$

Confirm Graph Using Geogebra https://www.geogebra.org/3d/njgcywym

2.3 Unit Tangent Vectors

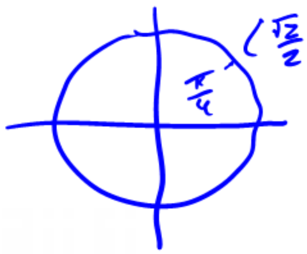
Definition 2.3.1

The **Unit Tangent Vector** is a vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ that has length 1.

Example 2.3.1 Find the unit tangent vector $\mathbf{T}(t)$ to the vector-valued function

$$\mathbf{r}(t) = 2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j} + \tan(t)\mathbf{k}$$

at $t = \frac{\pi}{4}$.



$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ $\frac{\pi}{4}$

$\vec{r}\,'(t) = 2\cos(t)\vec{i} - 2\sin(t)\vec{j} + \sec^2(t)\vec{k}$

$|\vec{r}\,'(t)| = \sqrt{4\cos^2(t) + 4\sin^2(t) + \sec^4(t)}$

$= \sqrt{4(\cos^2(t) + \sin^2(t)) + \sec^4(t)}$

$= \sqrt{4 + \sec^4(t)}$

$\vec{T}(t) = \dfrac{2\cos(t)\vec{i} - 2\sin(t)\vec{j} + \sec^2(t)\vec{k}}{\sqrt{4 + \sec^4(t)}}$

$\vec{T}(\frac{\pi}{4}) = (\sqrt{2}\,\vec{i} - \sqrt{2}\,\vec{j} + 2\vec{k}) / \sqrt{4+4}$

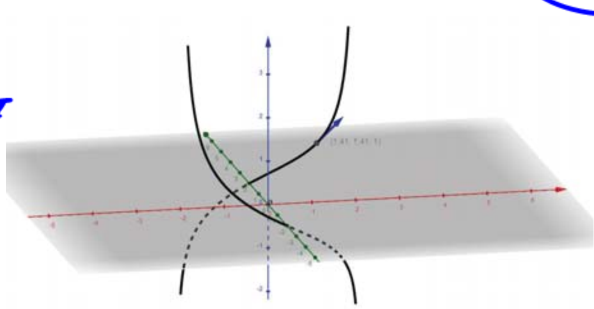


Figure 2.3.1: View Graph Using Geogebra https://www.geogebra.org/3d/mekypqct

$1/\frac{\sqrt{2}}{2}$

$= \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{2}}{2}$

$= \sqrt{2}$

$= \frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k}$ $\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{2}{4}} = \sqrt{\frac{4}{4}} = 1$ $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\sqrt{8} = 2\sqrt{2}$

Exercise 2.3.1 Find the unit tangent vector $\mathbf{T}(t)$ to the vector-valued function

$$\mathbf{r}(t) = \sin(2t)\mathbf{i} + 2t\mathbf{j} + \cos(t)\mathbf{k}$$

at $t = \frac{\pi}{2}$.

$$\vec{r}'(t) = 2\cos(2t)\vec{i} + 2\vec{j} - \sin(t)\vec{k}$$

$$|\vec{r}'(t)| = \sqrt{4\cos^2(2t) + 4 + \sin^2(t)}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$= \frac{2\cos(2t)\vec{i} + 2\vec{j} - \sin(t)\vec{k}}{\sqrt{4\cos^2(2t) + 4 + \sin^2(t)}}$$

$$\vec{T}(\pi/2) = \frac{-2\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{9}} = -\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$$

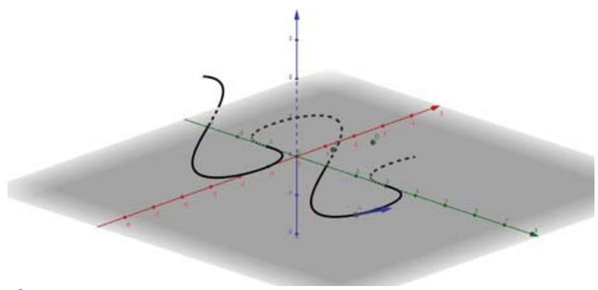


Figure 2.3.2: View Graph Using Geogebra
https://www.geogebra.org/3d/cqahgfcj

Example 2.3.2 Find the tangent vector and unit tangent vector to the curve with parametric equations

$$x = \cos(t), y = \sin(t), z = t$$

at the point (1,0,0) then determine the parametric equations for the tangent line at this point. Use technology to plot the space curve, tangent vector, unit tangent vector, and tangent line.

$t=0$

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\vec{T}(t) = \frac{\langle -\sin(t), \cos(t), 1 \rangle}{\sqrt{\sin^2(t) + \cos^2(t) + 1}}$$

$$= \langle -\tfrac{\sqrt{2}}{2}\sin(t), \tfrac{\sqrt{2}}{2}\cos(t), \tfrac{\sqrt{2}}{2} \rangle$$

$$\vec{T}_{\vec{r}(0)}(t) = \vec{r}(0) + t \cdot \vec{r}'(0)$$

$$= \langle 1,0,0 \rangle + t \cdot \langle 0,1,1 \rangle$$

$$= \langle 1, t, t \rangle$$

Geogebra instructions on next page.

To use Geogebra to plot the given objects, we start by inputting

(cos(t),sin(t),t)

in the first input box and change the starting t-value to -2π.
To generate the tangent vector, place the point by inputting into the second box $(1, 0, 0)$ and then input $a'(0)$, since $t = 0$ is where the point $(1, 0, 0)$ occurs, to get the tangent vector direction. To plot this as the tangent vector we enter

Vector(A,A+B)

so that Geogebra plots the vector from the point A in the direction $(0, 1, 1)$.
To plot the unit tangent vector, we then enter C = B/|B| to generate the unit tangent vector and, then in the next input box, enter

Vector(A,A+C)

. Finally, to plot the tangent line, we'll enter

$$\mathrm{X} = \mathrm{A} + t * \mathrm{C}$$

.

Confirm Graph Using Geogebra https://www.geogebra.org/3d/gd47kxjq

2.4 Angles of Intersection

Example 2.4.1 At what point do the curves $\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2\rangle$ and $\mathbf{r}_2(s) = \langle 3-s, s-2, s^2\rangle$ intersect? Find their angle of intersection.

$t = 3-s$

$1-t = s-2$

$1-(3-s) = s-2$

$-2+s = s-2$

$3+t^2 = s^2$

$3+(3-s)^2 = s^2$

$3+9-6s+\cancel{s^2} = \cancel{s^2}$

$12-6s = 0$

$12 = 6s$

$s = 2$

$t = 3-2$
$= 1$

$\vec{r}_1(1) = \langle 1, 0, 4\rangle$

$\vec{r}_2(2) = \langle 1, 0, 4\rangle$

$\vec{r}_1{}'(t) = \langle 1, -1, 2t\rangle$

$\vec{r}_1{}'(1) = \langle 1, -1, 2\rangle$

$\vec{r}_2{}'(s) = \langle -1, 1, 2s\rangle$

$\vec{r}_2{}'(2) = \langle -1, 1, 4\rangle$

$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$\cos(\theta) = \frac{\langle 1,-1,2\rangle \cdot \langle -1,1,4\rangle}{\sqrt{1+1+4}\cdot\sqrt{1+1+16}}$$

$$= \frac{-1-1+8}{\sqrt{6}\cdot\sqrt{18}}$$

$$= \frac{6}{\sqrt{3\cdot4\cdot9}}$$

$$= \frac{6}{2\cdot3\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$= \sqrt{3}/3$$

$$\theta = \cos^{-1}(\sqrt{3}/3) \approx 54.74^\circ$$

$6\cdot18$

$= 2\cdot3\cdot2\cdot3\cdot3$

$= 3\cdot4\cdot9$

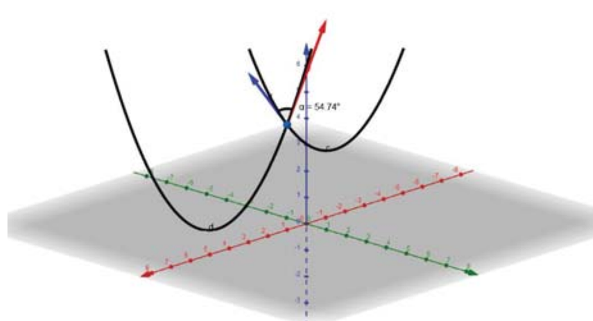


Figure 2.4.1: View Graph Using Geogebra
https://www.geogebra.org/3d/xbr3kubs

2.5 Derivative formulas

In general, the derivative formulas for real-valued functions are similar for vector-valued functions. Below are a few commonly used formulas for differentiation.

Theorem 2.5.1

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and c is a scalar, then

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)+\mathbf{v}(t)) = \mathbf{u}'(t)+\mathbf{v}'(t)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}(c\mathbf{u}(t)) = c\mathbf{u}'(t)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\cdot\mathbf{v}(t)) = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$

$\vec{a} \times \vec{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle$

These can be proven using the component form for each vector.

Example 2.5.1 Find $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\times\mathbf{v}(t))$ for $\mathbf{u}(t) = 2t\mathbf{i}+6t\mathbf{j}+t^2\mathbf{k}$ and $\mathbf{v}(t) = e^{-t}\mathbf{i}+e^{-t}\mathbf{j}+\mathbf{k}$.

Exercise 2.5.1 For vectors $\mathbf{u}$ and $\mathbf{v}$ in Example 2.5.1, find $\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$. How do the results compare with Example 2.5.1 and Theorem 2.5.1(d)?

2.6 Integration of Vector Valued Functions

Definite integrals of vector-valued functions may now be considered. Specifically, if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g, and h are integrable on $[a, b]$, then by definition,

$$\begin{aligned}\int_a^b \mathbf{r}(t)dt &= \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k} \\ &= \mathbf{R}(t)\big|_a^b \\ &= \mathbf{R}(b) - \mathbf{R}(a)\end{aligned}$$

from the Fundamental Theorem of Calculus such that $\mathbf{R}'(t) = \mathbf{r}(t)$.

Likewise, if $\mathbf{R}'(t) = \mathbf{r}(t)$, then every antiderivative of $\mathbf{r}(t)$ is of the form $\mathbf{R}(t) + \mathbf{C}$ for some constant vector $\mathbf{C}$. We can therefore write $\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}$ if $\mathbf{R}'(t) = \mathbf{r}(t)$

Example 2.6.1 Evaluate $\displaystyle\int_0^1 \left(\frac{4}{1+t^2}\mathbf{j} + \frac{2t}{1+t^2}\mathbf{k}\right) dt$.

Exercise 2.6.1 Evaluate $\displaystyle\int (\sec^2(t)\mathbf{i} + t(t^2+1)^3\mathbf{j} + t^2\ln(t)\mathbf{k})dt$.