LESSON 14

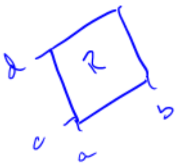
Iterated Integrals

Contents

14.1 Volumes via Double Integrals

$\frac{\partial}{\partial x}(2x^2+3xy+y^2) = 4x+3y$

If we consider past experience with partial differentiation, we hold all variables constant except the single variable we differentiate with respect to and then differentiate as usual. This 'similar' process will be used to integrate functions of several variables over a region by holding one variable constant and then integrating with respect to the other variable(s).



To illustrate this concept with double integral, consider $z = f(x, y)$ continuous over region $[a, b] \times [c, d]$. First we fix the x-variable and get the area under the curve $z = f(x_i, y)$. By thinking of x as a fixed constant in our expression for $f(x, y)$ and taking the integral with respect to y, we will get a function $A(x)$ which outputs the area under the curve parallel to the y-axis at the given x-value. By then integrating this function $z = A(x)$ over $a \leq x \leq b$, we total all of these areas to get the volume of ~~of~~ the solid between the xy-plane and the surface $z = f(x, y)$. Note that this argument can be repeated in a symmetric fashion by first holding y-constant to get a function $z = A(y)$ and then integrating with respect to y.

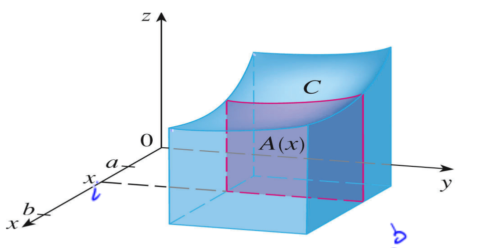


FIGURE 1

$\int_a^b A(x)\, x$

Figure 14.1.1: Fix $x\epsilon[a, b]$ for $\int_c^d f(x, y)\,\mathrm{d}y$

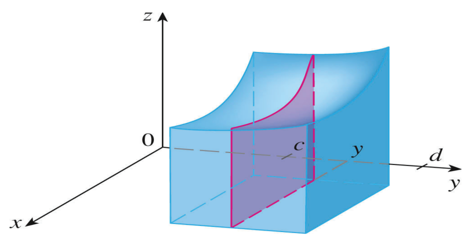


FIGURE 2

Figure 14.1.2: Fix $y\epsilon[c, d]$ for $\int_a^b f(x, y)\,\mathrm{d}x$

Thus the volume between the xy-plane and the surface $z = f(x, y)$ can be computed via the following theorem.

Theorem 14.1.1

Suppose that $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$. Then

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y\right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x, y)\,\mathrm{d}x\right) \mathrm{d}y.$$

This tells us how to compute double integrals by means ~~or~~ of two successive (or *iterated*) single-variable integrations, both of which we can compute using the Fundamental Theorem of Calculus, if f is sufficiently well behaved on R.

In practice, integration simply begins from the inside and moves outwards. That is, variable x is treated as a constant of integration for $\int_c^d f(x,y)\,\mathrm{d}y$ with respect to y, simplify those results, then treat y as a constant of integration with respect to x. Don't try and do too much at once.

Example 14.1.1 Calculate the iterated integral $\displaystyle\int_{-2}^{2}\left(\int_0^1 (xy+x^2)\,\mathrm{d}y\right)\mathrm{d}x$.

$$\int_{-2}^{2}\left(\int_0^1 xy+x^2\,dy\right)dx$$

$$=\int_{-2}^{2}\left(\frac{1}{2}xy^2+x^2\cdot y\right)\Big|_0^1\,dx$$

$$=\int_{-2}^{2}\frac{1}{2}x+x^2\,dx$$

$$=\int_{-2}^{2}\frac{1}{2}x+x^2\,dx$$

$$=\frac{1}{4}x^2+\frac{1}{3}x^3\Big|_{-2}^{2}$$

$$=1+\frac{8}{3}-\left(1-\frac{8}{3}\right)$$

$$=16/3=5\frac{1}{3}$$

Exercise 14.1.1 From the previous example 14.1.1, interchange the order to integrate by switching the x and y bounds and $\mathrm{d}x$ and $\mathrm{d}y$. That is, evaluate $\int_0^1\int_{-2}^{2}(xy+x^2)\,\mathrm{d}x\,\mathrm{d}y$.

$$\int_0^1\left(\int_{-2}^{2}xy+x^2\,dx\right)dy$$

$$=\int_0^1\left(\frac{1}{2}yx^2+\frac{1}{3}x^3\right)\Big|_{-2}^{2}dy$$

$$=\int_0^1 2y+\frac{8}{3}-\left(2y-\frac{8}{3}\right)dy$$

$$=\int_0^1 16/3\,dy$$

$$=\frac{16}{3}y\Big|_0^1$$

$$=16/3=5\frac{1}{3}$$

Example 14.1.2 Evaluate $\iint\limits_R x\cos(x+y)dA$ where $R = \{(x,y)|0 \le x \le \frac{\pi}{4}, 0 \le y \le \frac{\pi}{3}\}$.

$$\iint\limits_R x\cos(x+y)\,dA = \int_0^{\pi/3}\left(\int_0^{\pi/4} x\cos(x+y)\,dx\right)dy$$

Let $u = x$, $dv = \cos(x+y)dx$; $du = dx$, $v = \sin(x+y)$

$$= \int_0^{\pi/3}\left(x\sin(x+y)\Big|_0^{\pi/4} - \int_0^{\pi/4}\sin(x+y)\,dx\right)dy$$

$$= \int_0^{\pi/3}\left[\frac{\pi}{4}\sin\left(\frac{\pi}{4}+y\right) - 0 - \left(-\cos(x+y)\right)\Big|_0^{\pi/4}\right]dy$$

$$= \int_0^{\pi/3}\frac{\pi}{4}\sin\left(\frac{\pi}{4}+y\right) + \cos\left(\frac{\pi}{4}+y\right) - \cos(y)\,dy$$

$$= -\frac{\pi}{4}\cos\left(\frac{\pi}{4}+y\right) + \sin\left(\frac{\pi}{4}+y\right) - \sin(y)\Big|_0^{\pi/3}$$

$$= -\frac{\pi}{4}\cos(7\pi/12) + \sin(7\pi/12) - \frac{\sqrt{3}}{2} - \left(-\frac{\pi}{4}\cos(\pi/4) + \sin\left(\frac{\pi}{4}\right) - 0\right)$$

$$= -\frac{\pi}{4}\cos(7\pi/12) + \sin(7\pi/12) - \frac{\sqrt{3}}{2} + \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{2}}{2} \approx 0.151430$$

$$\int_0^{\pi/4}\int_0^{\pi/3} x\cos(x+y)\,dy\,dx$$

$$= \int_0^{\pi/4}\left(x\sin(x+y)\right)\Big|_0^{\pi/3}dx$$

$u = x$, $dv = \sin(x+c)dx$; $du = dx$, $v = -\cos(x+c)$

$$= \int_0^{\pi/4} x\sin(x+\pi/3) - x\sin(x)\,dx$$

$$= \left(-x\cos\left(x+\frac{\pi}{3}\right)\right)\Big|_0^{\pi/4} + \int_0^{\pi/4}\cos\left(x+\frac{\pi}{3}\right)dx - \left(\left(-x\cos(x)\right)\Big|_0^{\pi/4} + \int_0^{\pi/4}\cos(x)\,dx\right)$$

$$= -\left(\frac{\pi}{4}\cos(7\pi/12) - 0\right) + \left(\sin\left(x+\frac{\pi}{3}\right)\right)\Big|_0^{\pi/4} - \left(-\left(\frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) - 0\right) + \left(\sin(x)\right)\Big|_0^{\pi/4}\right)$$

$$= -\frac{\pi}{4}\cos\left(\frac{7\pi}{12}\right) + \sin(7\pi/12) - \sin\left(\frac{\pi}{3}\right) - \left(-\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\pi}{4}\cos\left(\frac{7\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) - \frac{\sqrt{3}}{2} + \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{2}}{2}$$

✓

14.2 Average Value

As a reminder from the previous section, the average z-value of a function $z = f(x,y)$ is given by

$$z_{ave} = \frac{1}{A(D)}\iint\limits_D f(x,y)dA.$$

Exercise 14.2.1 Find the average value of $f(x,y) = e^x\sqrt{x+e^y}$ over $D = [0,4]\times[0,2]$.

$$z_{ave} = \frac{1}{A(D)}\iint\limits_D f(x,y)\,dA$$

$$= \frac{1}{8}\int_0^2\int_0^4 e^x(x+e^y)^{1/2}\,dx\,dy$$

$$\approx 33.1276$$

$A(D) = 4\cdot 2 = 8$

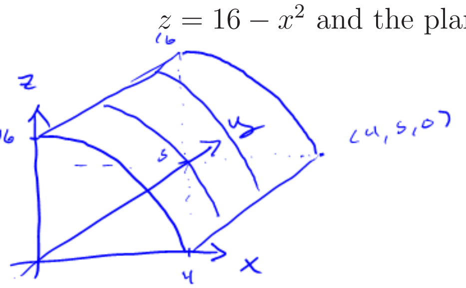
$$\int e^x(x+c)^{1/2}\,dx$$

$u = e^x$, $dv = (x+c)^{1/2}dx$; $du = e^x$, $v = \frac{2}{3}(x+c)^{3/2}$

$$= \frac{2}{3}e^x(x+c)^{3/2} - \frac{2}{3}\int e^x(x+c)^{3/2}dx$$

14.3 Volumes of Solids in Other Orientations

Example 14.3.1 Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$



$$V = \iint_{[0,4]\times[0,5]} 16 - x^2 \, dA$$

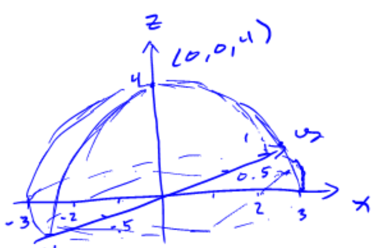
$$= \int_0^5 \int_0^4 16 - x^2 \, dx \, dy$$

$$= \int_0^5 16x - \frac{1}{3}x^3 \Big|_0^4 dy$$

$$= \int_0^5 64 - \frac{64}{3} \, dy$$

$$= \frac{128}{3} y \Big|_0^5 = \frac{640}{3} = 213\frac{1}{3}$$

Exercise 14.3.1 Find the volume of the solid lying under the elliptic paraboloid $x^2/9 + y^2 + z^2/4 = 1$ and above the rectangle $R = [-2, 2] \times [-.5, .5]$.



$$\frac{z}{4} = 1 - \frac{x^2}{9} - y^2$$

$$z = 4 - \frac{x^2}{9/4} - \frac{y^2}{1/4}$$

$\frac{x^2}{9} + y^2 + \frac{z}{4} = 1$ if $z = 0$: $\frac{x^2}{9} + y^2 = 1$

gives ellipse with x-rad = 3, y-rad = 1

pluggin in $(2, 0.5)$ for x + y:

$$z = 4 - \frac{4}{9/4} - \frac{1/4}{1/4}$$

$$= 4 - \frac{16}{9} - 1 = \frac{27}{9} - \frac{16}{9} = \frac{11}{9}$$

$$V = \int_{-1/2}^{1/2} \int_{-2}^{2} 4 - \frac{4x^2}{9} - 4y^2 \, dx \, dy$$

$$= \int_{-1/2}^{1/2} 4x - \frac{4x^3}{27} - 4y^2 x \Big|_{-2}^{2} dy$$

$$= \int_{-1/2}^{1/2} 8 - \frac{32}{27} - 8y^2 - \left(-8 + \frac{32}{27} + 8y^2\right) dy$$

$$= \int_{-1/2}^{1/2} 16 - \frac{64}{27} - 16y^2 \, dy$$

$$= 16y - \frac{64}{27}y - \frac{16}{3}y^3 \Big|_{-1/2}^{1/2}$$

$$= 8 - \frac{32}{27} - \frac{2}{3} - \left(-8 + \frac{32}{27} + \frac{2}{3}\right)$$

$$= 16 - \frac{64}{27} - \frac{36}{27} = 16 - \frac{100}{27} = \frac{332}{27} = 12\frac{8}{27}$$