LESSON 15

Double Integrals over General Regions

Contents

15.1 Type I Double Integrals

So far we've only looked at finding the volume under a surface over a rectangular region in x and y. In those situations we found constants for our bounds in both the x and y-directions and that we could switch wether we were integrating with respect to y first and x second or visa versa. That is, we found that integrating $f(x, y)$ over the region $[a, b] \times [c, d]$ is obtained by

$$\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

$y = -x^2 + 1$

$y = x^2 - 1$

However, if we want to find the volume under a surface and over a domain that is not rectangular but instead bounded by two equations in x and y, then we have to give much more care to the construction of our double integral. We look first at situations where it is most convenient to integrate with respect to y first and then with respect to x, which we refer to as **Type I Double Integrals**. We find that this occurs when we are integrating over ~~is described by~~ the region bounded between two functions $y = f(x)$ and $y = g(x)$ over some fixed x-interval. This is demonstrated in the following example.

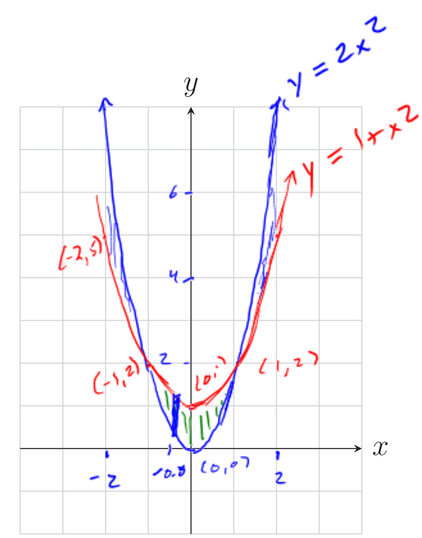
Example 15.1.1 Evaluate $\iint\limits_D (x + 2y)dA$ if D is the region bounded by $y_l = 2x^2$ and $y_h = 1 + x^2$.

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y \, dy\, dx = \int_{-1}^{1} \left(xy + y^2\right)\Big|_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \left(x+x^3+1+2x^2+x^4-(2x^3+4x^4)\right) dx$$

$$= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1 \, dx$$

$$= -\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \Big|_{-1}^{1} = \frac{32}{15}$$

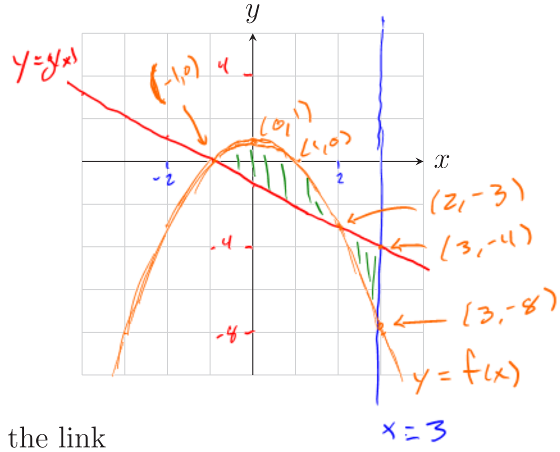


You may view a graph of this in geogebra by following the link https://www.geogebra.org/3d/cjbqpphz.

In the above example, we needed to find the intersections between the two functions in order to determine our bounds on x and these were the natural boundaries in the x-direction. The next example highlights some of the complications we can run into as we set up our double-integral.

Exercise 15.1.1 Set up, *but do not evaluate*, $\iint\limits_D (x+2y)dA$ if D is the region bounded by $f(x) = -x^2+1$, $g(x) = -x-1$, and $x = 3$.

$$V = \int_{-1}^{2}\int_{-x-1}^{-x^2+1}(x+2y)\,dy\,dx + \int_{2}^{3}\int_{-x^2+1}^{-x-1}(x+2y)\,dy\,dx$$



You may view a graph of this in geogebra by following the link https://www.geogebra.org/3d/txqv8z86.

15.2 Type II Double Integrals

A **Type II Double Integral**, on the other hand, is a situation where it is most convenient to integrate with respect to x first and then with respect to y. We find that this occurs when we are integrating over ~~is described by~~ the region bounded between two functions $x = f(y)$ and $x = g(y)$ over some fixed y-interval.

Example 15.2.1 Evaluate $\iint\limits_D (2x-y^2)dA$ if D is the region bounded by $x = (y-1)^2 - 1$ and $x = 2y$.

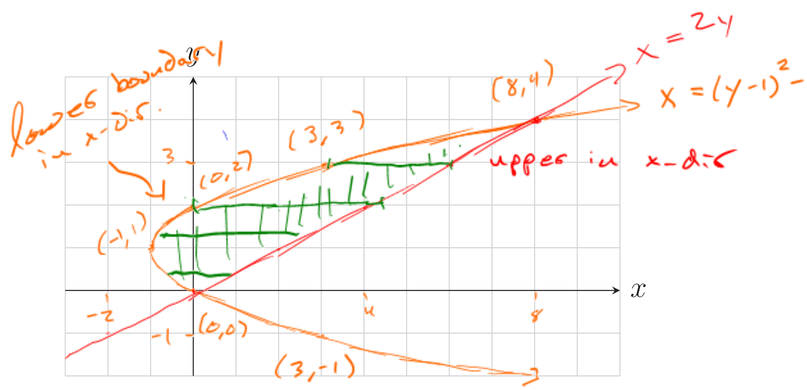
$$\int_0^4 \left(\int_{x_l=(y-1)^2-1}^{x_u=2y} (2x-y^2)\,dx \right) dy$$

$$(y-1)^2-1 = y^2-2y+1-1 = y^2-2y$$

$$= \int_0^4 \left(x^2 - y^2 x \right)\Big|_{(y-1)^2-1}^{2y} dy$$

$$= \int_0^4 (2y)^2 - y^2(2y) - \left((y^2-2y)^2 - y^2(y^2-2y)\right) dy$$

$$= \int_0^4 0\,dy = 0$$



You may view a graph of this in geogebra by following the link https://www.geogebra.org/3d/guecvshf.

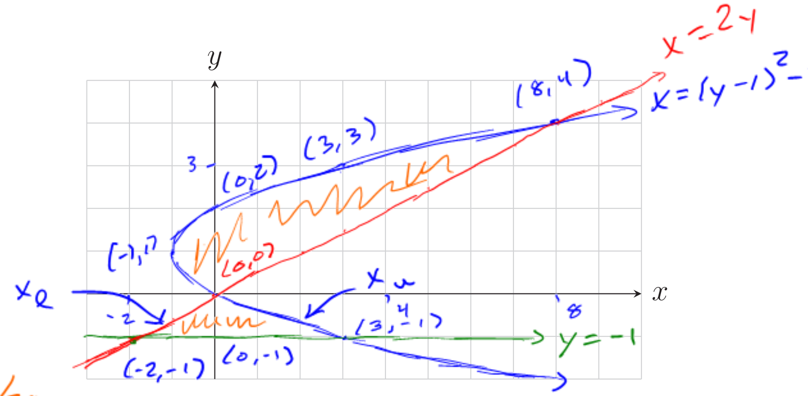
Exercise 15.2.1 Evaluate, $\iint\limits_D (2x - y^2)dA$ if D is the region bounded by $x = (y-1)^2 - 1$, $x = 2y$, and $y = -1$. Note that two of these boundaries are the same as the previous example.

$$V = \int_0^4 \int_{(y-1)^2-1}^{2y} 2x - y^2 \, dx\, dy + \int_{-1}^{0} \int_{2y}^{(y-1)^2-1} 2x - y^2 \, dx\, dy$$

$$= 0 + \int_{-1}^{0} (x^2 - y^2 x)\Big|_{2y}^{y^2-2y} dy$$

$$= \int_{-1}^{0} (y^2-2y)^2 - y^2(y^2-2y) - (4y^2 - 2y^3)\, dy$$

$$= \int_{-1}^{0} y^4 - y^3 \, dy = \frac{1}{5}y^5 - \frac{1}{4}y^4 \Big|_{-1}^{0} = -\left(-\frac{1}{5} - \frac{1}{4}\right) = 9/20$$



You may view a graph of this in geogebra by following the link https://www.geogebra.org/3d/d4st9acg.

15.3 Switching Between Type I and Type II Double Integrals

In every situation it is possible to use either a Type I or Type II integral, although it is often easier to choose one over the other. Let's go through this to highlight further how one can be easier than the other.

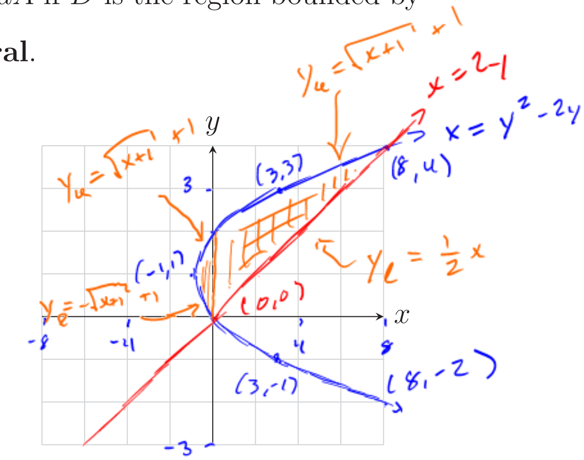
Example 15.3.1 Set up, but do not evaluate, $\iint\limits_D (2x - y^2)dA$ if D is the region bounded by $x = (y-1)^2 - 1$ and $x = 2y$ **as a Type I Double Integral**.

$x + 1 = (y-1)^2$ $\qquad y = \frac{1}{2}x$

$\pm\sqrt{x+1} = y - 1$

$y = \pm\sqrt{x+1} + 1$

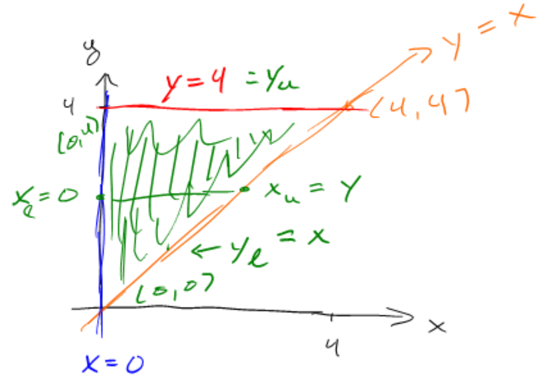
$$V = \int_{-1}^{0} \int_{y_l = -\sqrt{x+1}+1}^{y_u = \sqrt{x+1}+1} (2x - y^2)\, dy\, dx + \int_0^8 \int_{x/2}^{\sqrt{x+1}+1} (2x - y^2)\, dy\, dx = 0$$



Exercise 15.3.1 Set up the iterated integral $\iint\limits_D y^2 e^{xy} dA$, where D is bounded by $y = x$, $y = 4$, and $x = 0$ for both orders of integration.

Type I: Type II:

$$\int_0^4\int_x^4 y^2 e^{xy}\,dy\,dx = \int_0^4\int_0^y y^2 e^{xy}\,dx\,dy$$



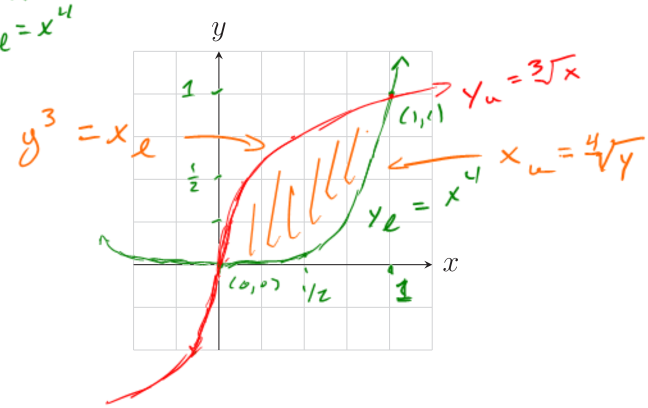
Now lets take a look at how we may interpret a double integral whose bounds are given as the volume of a solid and then use this interpretation to change the order of integration.

$\leftarrow y_u = x^{1/3} = \sqrt[3]{x}$ $y_u = \sqrt[3]{x}$

Example 15.3.2 Change the order of integration of $\int\limits_0^1 \int\limits_{x^4}^{x^{1/3}} f(x, y) dy dx$.

$y_\ell = x^4$

$$\int_0^1\int_{x^4}^{x^{1/3}} f(x,y)\,dy\,dx = \int_0^1\left(\int_{y^3}^{\sqrt[4]{y}} f(x,y)\,dx\right)dy$$



15.4 Volumes of Solids

So far in this lesson we've described a double integral with a region over which to integrate and have seen answers that are both postive and negative depending on how much of the enclosed volume is below the xy-plane and how much is above the xy-plane since integrals give volumes under the xy-plane as negative volumes. Now we look at exclusively positive volumes that are described via the equations of the surfaces which bound them and then set up the integral ourselves.

Example 15.4.1 Determine the volume of the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, and $z = 0$ in the first octant.

$z = +\sqrt{4-y^2}$

$$V = \int_0^4 \int_{\frac{1}{2}x}^{2} \sqrt{4-y^2}\, dy\, dx$$

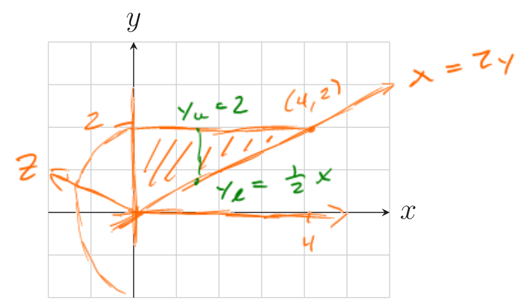
Set $y = 2\sin(u)$

$dy = 2\cos(u)du$

$y = 2 = 2\sin(u)$

$\frac{1}{2}x = 2\sin(u)$

$u = \sin^{-1}(\frac{1}{4}x)$



$$= \int_0^4 \int_{\sin^{-1}(\frac{1}{4}x)}^{\pi/2} 2\cos(u)\sqrt{4-4\sin^2(u)}\, du\, dx$$

$$= \int_0^4 \int_{\sin^{-1}(\frac{1}{4}x)}^{\pi/2} 4\cos^2(u)\, du\, dx$$

$$= \int_0^4 \int_{\sin^{-1}(\frac{1}{4}x)}^{\pi/2} 2\cos(2u) + 2\, du\, dx$$

$$= \int_0^4 \sin(2u) + 2u \Big|_{\sin^{-1}(\frac{1}{4}x)}^{\pi/2} dx$$

$$= \int_0^4 0 + \pi - \sin(2\sin^{-1}(\tfrac{1}{4}x)) - 2\sin^{-1}(\tfrac{1}{4}x)\, dx$$

$$= 16/3$$

$\sqrt{4-4\sin^2(u)} = \sqrt{4(1-\sin^2(u))} = \sqrt{4\cos^2(u)}$

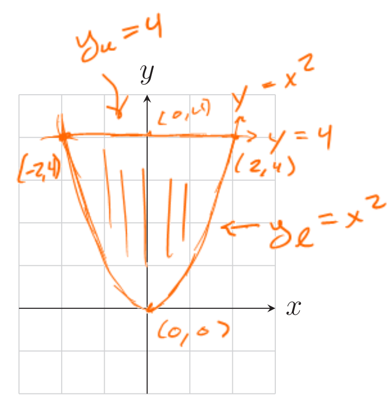
$= 2\cos(u)$

$\cos^2(u) = \frac{\cos(2u)+1}{2}$

You may view a graph of this in geogebra by following the link https://www.geogebra.org/3d/tjxt6xre.

Exercise 15.4.1 Find the volume of the solid bounded by the cylinders $z = x^2$, $y = x^2$, and the planes $z = 0$ and $y = 4$.

$$V = \int_{-2}^{2} \int_{x^2}^{4} x^2\, dy\, dx$$



$$= \int_{-2}^{2} x^2 y \Big)_{x^2}^{4} dx$$

$$= \int_{-2}^{2} 4x^2 - x^4\, dx$$

$$= \frac{4}{3}x^3 - \frac{1}{5}x^5 \Big|_{-2}^{2}$$

$$= \frac{32}{3} - \frac{32}{5} - \left(-\frac{32}{3} + \frac{32}{5}\right)$$

$$= \frac{64}{3} - \frac{64}{5}$$

$$= \frac{320-192}{15} = \frac{128}{15} = 8\tfrac{8}{15}$$

You may view a graph of this in geogebra by following the link https://www.geogebra.org/3d/nbvt83kc.