LESSON 18

Surface Area

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18.1 Introduction and Formulation

Given a surface $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ with domain $D = [a,b] \times [c,d]$, it is of interest to compute the surface area of the surface over the given domain. In order to compute this area we will begin by cutting the domain into sub-rectangles, $R_{i,j} = [u_i, u_{i+1}] \times [v_j, v_{j+1}]$ with area $\Delta A = \Delta u \cdot \Delta v$. By doing so we simultaneously cut the surface into "patches," $S_{i,j} = [\mathbf{r}(u_i, v_j), \mathbf{r}(u_{i+1}, v_j)] \times [\mathbf{r}(u_i, v_j), \mathbf{r}(u_i, v_{j+1})]$

Now, fixing v_j, Let's look at $\Delta\mathbf{r}$ as we move by Δu:

$$\Delta\mathbf{r}_u = \mathbf{r}(u_i, v_j) - \mathbf{r}(u_i + \Delta u, v_j) \approx \mathbf{r}_u(u_i, v_j) \cdot \Delta u.$$

And, similarly, if we fix u_i and move by Δv we get

$$\Delta\mathbf{r}_v \approx \mathbf{r}_v(u_i, v_j) \cdot \Delta v.$$

Then the area of $S_{i,j}$ can be approximated by the area of the parallelogram determined by $\Delta u \cdot \mathbf{r}_u(u_i, v_j)$ and $\Delta v \cdot \mathbf{r}_v(u_i, v_j)$ and since parallelograms area is equal to the magnitude of the cross product of the vectors which determin them we get:

$$A_{i,j} \approx |\Delta u \cdot \mathbf{r}_u(u_i, v_j) \times \Delta v \cdot \mathbf{r}_v(u_i, v_j)| = |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|\Delta u \Delta v = |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|\Delta A$$

Summing the areas of all the patches and letting Δu and Δv go to zero we get the total area of the surface to be

$$\begin{aligned} A(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|\Delta A \\ &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA \end{aligned}$$

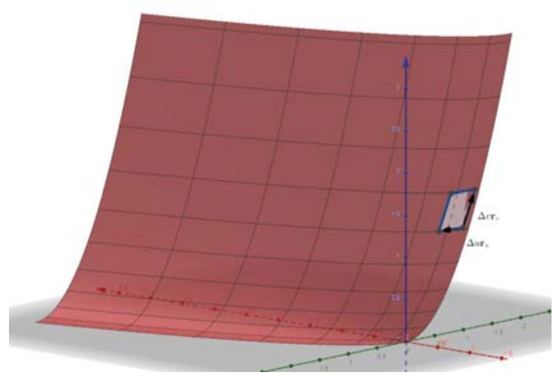

Figure 18.1.1: Surface Area of Approximating Parallelogram
View in Geogebra: https://www.geogebra.org/3d/r2v9pqfa

Example 18.1.1 Given $\mathbf{r}(u,v) = \langle u+v, 2-3u, 1+u-v\rangle$, find the surface area when $D = [0,2] \times [-1,1]$.

Exercise 18.1.1 Find the surface area of the helicoid (spiral ramp) with vector equation $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$ over $D = [0,1] \times [0,\pi]$.

18.2 Parameterizing a Surface with a Given Domain to Find the Surface Area

Example 18.2.1 Find the surface area of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Exercise 18.2.1 Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies inside the cylinder $x^2 + y^2 = 4$.

18.3 Determining the Surface Area for a Function in Two-Variables Over a Given Domain

Suppose we have a surface given by $z = f(x, y)$ and we want to find the surface area over a domain, D. We can parameterize the surface as

$$\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$$

Then $\mathbf{r}_x(x, y) = \langle 1, 0, f_x\rangle$ and $\mathbf{r}_y(x, y) = \langle 0, 1, f_y\rangle$. Then

$$|\mathbf{r}_x \times \mathbf{r}_y| = |\langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle| = |\langle -f_x, -f_y, 1\rangle| = \sqrt{1 + f_x^2 + f_y^2}$$

and thus the surface area can be computed via

$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\,dA$$

Example 18.3.1 Find the area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$.

Exercise 18.3.1 Determine the area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.