LESSON 18

Surface Area

Contents

18.1 Introduction and Formulation

u-interval v-interval

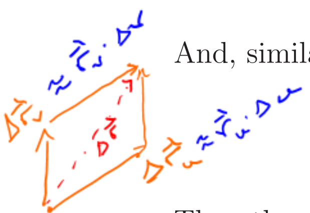
Given a surface $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ with domain $D = [a,b] \times [c,d]$, it is of interest to compute the surface area of the surface over the given domain. In order to compute this area we will begin by cutting the domain into sub-rectangles, $R_{i,j} = [u_i, u_{i+1}] \times [v_j, v_{j+1}]$ with area $\Delta A = \Delta u \cdot \Delta v$. By doing so we simultaneously cut the surface into "patches," $S_{i,j} = [\mathbf{r}(u_i, v_j), \mathbf{r}(u_{i+1}, v_j)] \times [\mathbf{r}(u_i, v_j), \mathbf{r}(u_i, v_{j+1})]$

Now, fixing v_j, Let's look at $\Delta\mathbf{r}$ as we move by Δu:

partial derivative with respect to u.

$$\Delta\mathbf{r}_u = \mathbf{r}(u_i, v_j) - \mathbf{r}(u_i + \Delta u, v_j) \approx \mathbf{r}_u(u_i, v_j) \cdot \Delta u.$$

$$\vec{r}_u = \lim_{\Delta u \to 0} \frac{\vec{r}(u_i + \Delta u, v_j) - \vec{r}(u_i, v_j)}{\Delta u}$$



And, similarly, if we fix u_i and move by Δv we get

$$\Delta\mathbf{r}_v \approx \mathbf{r}_v(u_i, v_j) \cdot \Delta v.$$

Then the area of $S_{i,j}$ can be approximated by the area of the parallelogram determined by $\Delta u \cdot \mathbf{r}_u(u_i, v_j)$ and $\Delta v \cdot \mathbf{r}_v(u_i, v_j)$ and since parallelograms area is equal to the magnitude of the cross product of the vectors which determin them we get: $A_{\square} = |\vec{a} \times \vec{b}|$

$$A_{i,j} \approx |\Delta u \cdot \mathbf{r}_u(u_i, v_j) \times \Delta v \cdot \mathbf{r}_v(u_i, v_j)| = |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|\Delta u \Delta v = |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|\Delta A$$

Summing the areas of all the patches and letting Δu and Δv go to zero we get the total area of the surface to be

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} |\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)|\Delta A$$
$$= \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

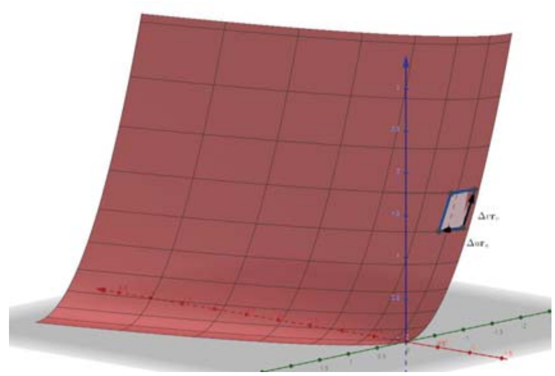


Figure 18.1.1: Surface Area of Approximating Parallelogram
View in Geogebra:
https://www.geogebra.org/3d/r2v9pqfa

Example 18.1.1 Given $\mathbf{r}(u,v) = \langle u+v, 2-3u, 1+u-v\rangle$, find the surface area when $D = [0,2]\times[-1,1]$.

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$$

$\vec{r}_u = \langle 1, -3, 1\rangle$ $\quad \vec{r}_v = \langle 1, 0, -1\rangle$

$$|\vec{r}_u \times \vec{r}_v| = \left| \begin{pmatrix}1\\-3\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\-1\end{pmatrix}\right| = |\langle 3, 2, 3\rangle|$$
$$= \sqrt{9+4+9} = \sqrt{22}$$

$$= \int_{-1}^{1}\int_0^2 \sqrt{22}\, du\, dv$$
$$= \int_{-1}^{1} \sqrt{22}\cdot u\Big|_0^2\, dv$$
$$= \int_{-1}^{1} 2\sqrt{22}\, dv$$
$$= 2\sqrt{22}\, v\Big|_{-1}^{1}$$
$$= 2\sqrt{22} - (-2\sqrt{22})$$
$$= 4\sqrt{22}$$

Exercise 18.1.1 Find the surface area of the helicoid (spiral ramp) with vector equation $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$ over $D = [0,1]\times[0,\pi]$.

$\vec{r}_u = \langle \cos(v), \sin(v), 0\rangle$

$\vec{r}_v = \langle -u\sin(v), u\cos(v), 1\rangle$

$\vec{r}_u \times \vec{r}_v = \langle \sin(v), -\cos(v), u\cos^2(v) + u\sin^2(v)\rangle$

$= \langle \sin(v), -\cos(v), u\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2(v) + \cos^2(v) + u^2}$

$= \sqrt{1+u^2}$

$$A(S) = \int_0^{\pi}\int_0^1 \sqrt{1+u^2}\, du\, dv$$

$u = \tan\theta$
$du = \sec^2\theta\, d\theta$

$$= \int_0^{\pi}\int_0^{\pi/4} \sqrt{1+\tan^2\theta}\,\sec^2\theta\, d\theta\, dv$$

$$= \int_0^{\pi}\int_0^{\pi/4} \sec^3\theta\, d\theta\, dv$$

$w = \sec\theta \quad dv = \sec^2\theta\, d\theta$
$dw = \sec\theta\tan\theta\, d\theta$
$v = \tan\theta$

$$\int_0^{\pi}\int_0^{\pi/4} \sec^3(\theta)\, d\theta\, dv = \int_0^{\pi}\left(\sec\theta\tan\theta\Big|_0^{\pi/4} - \int_0^{\pi/4}\sec\theta\tan^2\theta\, d\theta\right) dv$$

$$\int_0^{\pi}\int_0^{\pi/4} \sec^3(\theta)\, d\theta\, dv = \int_0^{\pi}\left(\sqrt{2} - \int_0^{\pi/4} -\sec\theta + \sec^3\theta\, d\theta\right) dv$$

$$\int_0^{\pi}\int_0^{\pi/4} \sec^3(\theta)\, d\theta\, dv = \int_0^{\pi}\sqrt{2}\, dv + \int_0^{\pi}\int_0^{\pi/4}\sec\theta\, d\theta\, dv - \int_0^{\pi}\int_0^{\pi/4}\sec^3\theta\, d\theta\, dv$$

$$2\int_0^{\pi}\int_0^{\pi/4} \sec^3(\theta)\, d\theta\, dv = \sqrt{2}\cdot\pi + \int_0^{\pi}\ln|\sec(\theta) + \tan(\theta)|\Big|_0^{\pi/4} dv$$

$$\Rightarrow \int_0^{\pi}\int_0^{\pi/4} \sec^3(\theta)\, d\theta\, dv = \frac{\sqrt{2}}{2}\pi + \ln(\sqrt{2}+1)\cdot\pi/2$$

$$\approx 3.6059$$

18.2 Parameterizing a Surface with a Given Domain to Find the Surface Area

Example 18.2.1 Find the surface area of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

$y = r\cos\theta \qquad z = r\sin\theta \qquad x = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$r^2 = 9$

$r = 3$

$\vec{S}(r,\theta) = \langle r^2, r\cos\theta, r\sin\theta\rangle \qquad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi$

$\vec{S}_r = \langle 2r, \cos\theta, \sin\theta\rangle$

$\vec{S}_\theta = \langle 0, -r\sin\theta, r\cos\theta\rangle$

$\vec{S}_r \times \vec{S}_\theta = \langle r\cos^2\theta + r\sin^2\theta, -2r^2\cos\theta, -2r^2\sin\theta\rangle$

$|\vec{S}_r \times \vec{S}_\theta| = \sqrt{r^2 + 4r^4\cos^2\theta + 4r^4\sin^2\theta}$

$= \sqrt{r^2 + 4r^4}$

$= r\sqrt{1 + 4r^2}$

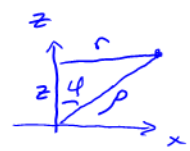
$A(S) = \int_0^{2\pi}\int_0^3 r\sqrt{1+4r^2}\,dr\,d\theta \qquad u = 1+4r^2, \; du = 8r\,dr$

$= \int_0^{2\pi} \frac{1}{8}\int_1^{37} \sqrt{u}\,du\,d\theta$

$= \int_0^{2\pi} \frac{1}{8}\cdot\frac{2}{3}u^{3/2}\Big|_1^{37} d\theta$

$= \int_0^{2\pi} \frac{1}{12}(37^{3/2} - 1)\,d\theta = \frac{\pi}{6}(37^{3/2} - 1)$

Exercise 18.2.1 Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies inside the cylinder $x^2 + y^2 = 4$. ← restraint when $r = 2$



$\vec{S}(\theta,\varphi) = \langle 3\sin(\varphi)\cos(\theta), 3\sin(\varphi)\sin(\theta), 3\cos(\varphi)\rangle$

$\vec{S}_\theta = \langle -3\sin(\varphi)\sin(\theta), 3\sin(\varphi)\cos(\theta), 0\rangle$

$\vec{S}_\varphi = \langle 3\cos(\varphi)\cos(\theta), 3\cos(\varphi)\sin(\theta), -3\sin(\varphi)\rangle$

$\vec{S}_\theta \times \vec{S}_\varphi = \langle -9\sin^2(\varphi)\cos(\theta), -9\sin^2(\varphi)\sin(\theta), -9\sin(\varphi)\cos(\varphi)\rangle$

$|\vec{S}_\theta \times \vec{S}_\varphi| = 9\sqrt{\sin^4(\varphi) + \sin^2(\varphi)\cos^2(\varphi)}$

$= 9\sin(\varphi)$

$r = 3\sin(\varphi)$

$2 = 3\sin(\varphi)$

$\Rightarrow \varphi = \sin^{-1}(2/3)$

$A(S) = 9\int_0^{2\pi}\int_0^{\sin^{-1}(2/3)} \sin(\varphi)\,d\varphi\,d\theta$

$= 9\int_0^{2\pi} d\theta \cdot \left(-\cos(\varphi)\right)\Big|_0^{\sin^{-1}(2/3)}$

$= 18\pi\left(1 - \cos(\sin^{-1}(2/3))\right)$

$= 18\pi\left(1 - \sqrt{5}/3\right)$

18.3 Determining the Surface Area for a Function in Two-Variables Over a Given Domain

Suppose we have a surface given by $z = f(x, y)$ and we want to find the surface area over a domain, D. We can parameterize the surface as

$$\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$$

Then $\mathbf{r}_x(x, y) = \langle 1, 0, f_x\rangle$ and $\mathbf{r}_y(x, y) = \langle 0, 1, f_y\rangle$. Then

$$|\mathbf{r}_x \times \mathbf{r}_y| = |\langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle| = |\langle -f_x, -f_y, 1\rangle| = \sqrt{1 + f_x^2 + f_y^2}$$

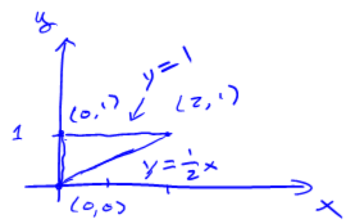
and thus the surface area can be computed via

$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\,dA$$

Example 18.3.1 Find the area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$.

$f(x,y) = 1 + 3x + 2y^2$

$f_x = 3 \qquad f_y = 4y$



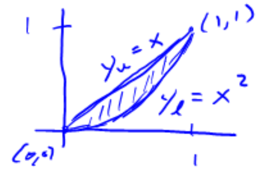
$$A(S) = \int_0^2 \int_{\frac{1}{2}x}^{1} \sqrt{1 + 9 + 16y^2}\,dy\,dx$$

$$= \frac{1}{12}\left(13\sqrt{26} - 5\sqrt{10}\right)$$

$$\approx 4.20632$$

Exercise 18.3.1 Determine the area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.

$f(x,y) = (x^2+y^2)^{1/2}$



$f_x = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x$ $\quad f_y = \frac{y}{\sqrt{x^2+y^2}}$

$= \frac{x}{\sqrt{x^2+y^2}}$

$A(S) = \int_0^1 \int_{x^2}^{x} \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dy\, dx$

$= \int_0^1 \int_{x^2}^{x} \sqrt{2}\, dy\, dx$

$= \int_0^1 \sqrt{2}\,x - \sqrt{2}\,x^2\, dx$

$= \frac{\sqrt{2}}{2} x^2 - \frac{\sqrt{2}}{3} x^3 \Big|_0^1$

$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{3}$