Carbon dating, which relies on the exponential decay of C^{14} relative to C^{12}, allows for the determination of the age of these cave paintings.
(© *John Mitchell/Alamy*)

5 THE INTEGRAL

The basic problem in integral calculus is finding the area under a curve. You may wonder why calculus deals with two seemingly unrelated topics: tangent lines on the one hand and areas on the other. One reason is that both are computed using limits. A deeper connection is revealed by the Fundamental Theorem of Calculus, discussed in Sections 5.4 and 5.5. This theorem expresses the "inverse" relationship between integration and differentiation. It plays a truly fundamental role in nearly all applications of calculus, both theoretical and practical.

5.1 Approximating and Computing Area

Why might we be interested in the area under a graph? Consider an object moving in a straight line with *constant velocity* v (assumed positive). The distance traveled over a time interval $[t_1, t_2]$ is equal to $v\Delta t$, where $\Delta t = (t_2 - t_1)$ is the time elapsed. This is the well-known formula

$$\text{Distance traveled} = \overbrace{\text{velocity} \times \text{time elapsed}}^{v\Delta t} \qquad (1)$$

Because v is constant, the graph of velocity is a horizontal line (Figure 1) and $v\Delta t$ is equal to the area of the rectangular region under the graph of velocity over $[t_1, t_2]$. So we can write Eq. (1) as

$$\text{Distance traveled} = \text{area under the graph of velocity over } [t_1, t_2] \qquad (2)$$

There is, however, an important difference between these two equations: Eq. (1) makes sense only if velocity v is constant, whereas Eq. (2) is correct *even if the velocity changes with time* (we will prove this in Section 5.6). Thus, the advantage of expressing distance traveled as an area is that it enables us to deal with much more general types of motion.

To see why Eq. (2) might be true in general, let's consider the case where velocity changes over time but is constant on intervals. In other words, we assume that the object's velocity changes abruptly from one interval to the next as in Figure 2. The distance traveled over each time interval is equal to the area of the rectangle above that interval, so the total distance traveled is the sum of the areas of the rectangles. In Figure 2,

$$\text{Distance traveled over } [0, 8] \text{ s} = \underbrace{10 + 15 + 30 + 10}_{\text{Sum of areas of rectangles}} = 65 \text{ m}$$

Our strategy when velocity changes continuously (Figure 3) is to *approximate* the area under the graph by sums of areas of rectangles and then pass to a limit. This idea leads to the concept of an integral.

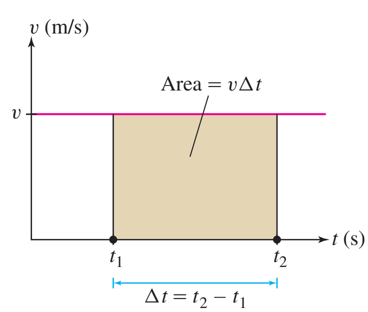


FIGURE 1 The rectangle has area $v\Delta t$, which is equal to the distance traveled.

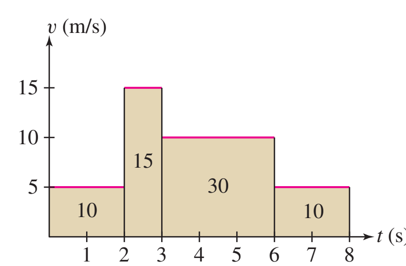


FIGURE 2 Distance traveled equals the sum of the areas of the rectangles.

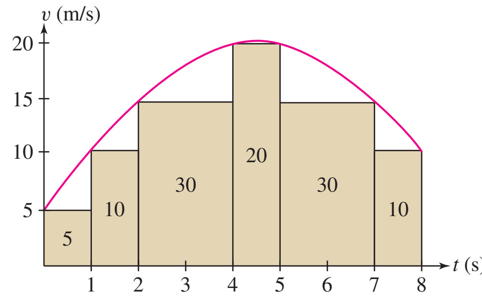


FIGURE 3 Distance traveled is equal to the area under the graph. It is *approximated* by the sum of the areas of the rectangles.

Approximating Area by Rectangles

Our goal is to compute the area under the graph of a function f. In this section, we assume that f is continuous and *positive*, so that the graph of f lies above the x-axis (Figure 4). The first step is to approximate the area using rectangles.

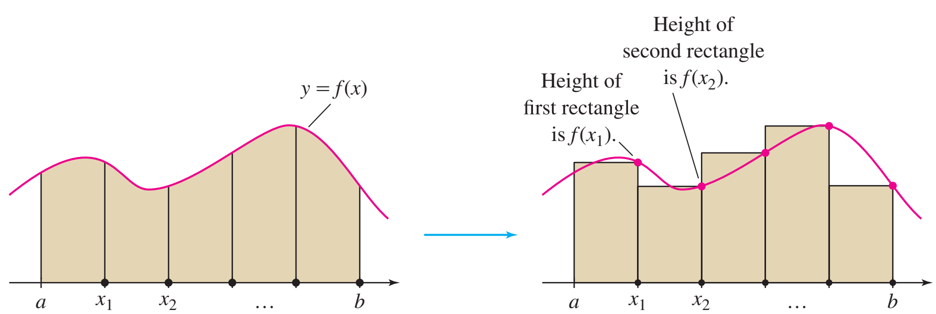
Recall the two-step procedure for finding the slope of the tangent line (the derivative): First approximate the slope using secant lines and then compute the limit of these approximations. In Integral Calculus, there are also two steps:

- *First, approximate the area under the graph using rectangles.*
- *Then compute the exact area (the integral) as the limit of these approximations.*

To begin, choose a whole number N and divide $[a, b]$ into N subintervals of equal width, as in Figure 4(A). The full interval $[a, b]$ has width $b - a$, so each subinterval has width $\Delta x = (b - a)/N$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x,\ \ x_2 = a + 2\Delta x,\ \ \ldots,\ \ x_{N-1} = a + (N-1)\Delta x,\ \ x_N = a + N\Delta x$$

Note that the last right endpoint is $x_N = b$ because $a + N\Delta x = a + N((b - a)/N) = b$. Next, as in Figure 4(B), construct, above each subinterval, a rectangle whose height is the value of $f(x)$ at the *right endpoint* of the subinterval.



(A) Divide $[a, b]$ into N subintervals, each of width Δx.

(B) Construct right-endpoint rectangles.

FIGURE 4

The sum of the areas of these rectangles provides an *approximation* to the area under the graph. The first rectangle has base Δx and height $f(x_1)$, so its area is $f(x_1)\Delta x$. Similarly, the second rectangle has height $f(x_2)$ and area $f(x_2)\,\Delta x$, etc. The sum of the areas of the rectangles is denoted R_N and is called the **Nth right-endpoint approximation**:

$$R_N = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_N)\Delta x$$

Factoring out Δx, we obtain the formula

$$R_N = \Delta x\left(f(x_1) + f(x_2) + \cdots + f(x_N)\right)$$

To summarize,

$$a = \textit{left endpoint of interval } [a, b]$$

$$b = \textit{right endpoint of interval } [a, b]$$

$$N = \textit{number of subintervals in } [a, b]$$

$$\Delta x = \frac{b - a}{N}$$

In words: *R_N is equal to Δx times the sum of the function values at the right endpoints of the subintervals.*

■ **EXAMPLE 1** Calculate R_4 and R_6 for $f(x) = x^2$ on the interval $[1, 3]$.

Solution

Step 1. **Determine Δx and the right endpoints.**

To calculate R_4, divide $[1, 3]$ into four subintervals of width $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. The right endpoints are the numbers $x_j = a + j\Delta x = 1 + j\left(\frac{1}{2}\right)$ for $j = 1, 2, 3, 4$. They are spaced at intervals of $\frac{1}{2}$ beginning at $\frac{3}{2}$, so, as we see in Figure 5(A), the right endpoints are $\frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}$.

Step 2. **Calculate Δx times the sum of function values.**

R_4 is Δx times the sum of the function values at the right endpoints:

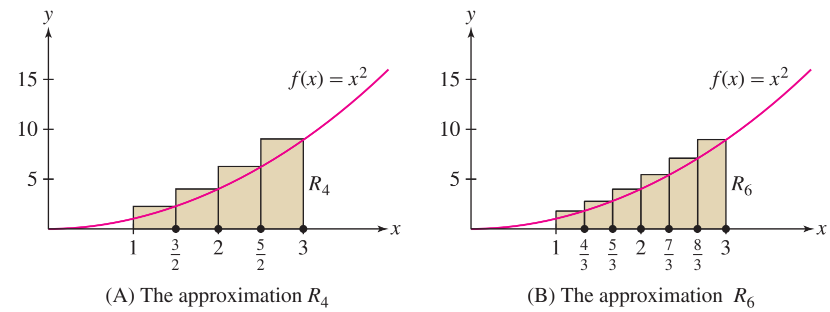
$$R_4 = \frac{1}{2}\left(f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{6}{2}\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{6}{2}\right)^2\right) = \frac{43}{4} = 10.75$$

R_6 is similar: $\Delta x = \frac{3-1}{6} = \frac{1}{3}$, and the right endpoints are spaced at intervals of $\frac{1}{3}$ beginning at $\frac{4}{3}$ and ending at 3, as in Figure 5(B). Thus,

$$R_6 = \frac{1}{3}\left(f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right) + f\left(\frac{6}{3}\right) + f\left(\frac{7}{3}\right) + f\left(\frac{8}{3}\right) + f\left(\frac{9}{3}\right)\right)$$

$$= \frac{1}{3}\left(\frac{16}{9} + \frac{25}{9} + \frac{36}{9} + \frac{49}{9} + \frac{64}{9} + \frac{81}{9}\right) = \frac{271}{27} \approx 10.037$$ ■



DF **FIGURE 5**

Summation Notation

Summation notation is a standard notation for writing sums in compact form. The sum of numbers $a_m, \ldots, a_n$ $(m \le n)$ is denoted

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$

The Greek letter $\sum$ (capital sigma) stands for "sum," and the notation $\sum_{j=m}^{n}$ tells us to start the summation at $j = m$ and end it at $j = n$. For example,

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

In this summation, the jth term is $a_j = j^2$. We refer to j^2 as the **general term**. The letter j is called the **summation index**. It is also referred to as a **dummy variable** because any other letter can be used instead. For example,

$$\sum_{k=4}^{6} (k^2 - 2k) = \overbrace{(4^2 - 2(4))}^{k=4} + \overbrace{(5^2 - 2(5))}^{k=5} + \overbrace{(6^2 - 2(6))}^{k=6} = 47$$

$$\sum_{m=7}^{9} 1 = 1 + 1 + 1 = 3 \qquad \text{(because } a_7 = a_8 = a_9 = 1\text{)}$$

The usual commutative, associative, and distributive laws of addition give us the following rules for manipulating summations.

Linearity of Summations

- $$\sum_{j=m}^{n}(a_j+b_j)=\sum_{j=m}^{n}a_j+\sum_{j=m}^{n}b_j$$
- $$\sum_{j=m}^{n}Ca_j=C\sum_{j=m}^{n}a_j \qquad (C \text{ any constant})$$
- $$\sum_{j=1}^{n}C=nC \qquad (C \text{ any constant and } n\ge 1)$$

For example,

$$\sum_{j=3}^{5}(j^2+j)=(3^2+3)+(4^2+4)+(5^2+5)$$

is equal to

$$\sum_{j=3}^{5}j^2+\sum_{j=3}^{5}j=\left(3^2+4^2+5^2\right)+\left(3+4+5\right)$$

Linearity can be used to write a single summation as a sum of several summations. For example,

$$\begin{aligned}\sum_{k=0}^{100}(7k^2-4k+9)&=\sum_{k=0}^{100}7k^2+\sum_{k=0}^{100}(-4k)+\sum_{k=0}^{100}9\\&=7\sum_{k=0}^{100}k^2-4\sum_{k=0}^{100}k+9\sum_{k=0}^{100}1\end{aligned}$$

It is convenient to use summation notation when working with area approximations. For example, R_N is a sum with general term $f(x_j)$:

$$R_N=\Delta x\big[f(x_1)+f(x_2)+\cdots+f(x_N)\big]$$

The summation extends from $j=1$ to $j=N$, so we can write R_N concisely as

$$R_N=\Delta x\sum_{j=1}^{N}f(x_j)$$

We shall make use of two other rectangular approximations to area: the left-endpoint and the midpoint approximations. Divide $[a,b]$ into N subintervals as before. In the **left-endpoint approximation** L_N, the heights of the rectangles are the values of $f(x)$ at the left endpoints [Figure 6(A)]. These left endpoints are

REMINDER

$$\Delta x=\frac{b-a}{N}$$

$$x_0=a,\ \ x_1=a+\Delta x,\ \ x_2=a+2\Delta x,\ \ \ldots,\ \ x_{N-1}=a+(N-1)\Delta x$$

and the sum of the areas of the left-endpoint rectangles is

$$L_N=\Delta x\Big(f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{N-1})\Big)$$

Note that both R_N and L_N have general term $f(x_j)$, but the sum for L_N runs from $j=0$ to $j=N-1$ rather than from $j=1$ to $j=N$:

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

In the **midpoint approximation** M_N, the heights of the rectangles are the values of $f(x)$ at the midpoints of the subintervals rather than at the endpoints. As we see in Figure 6(B), the midpoints are

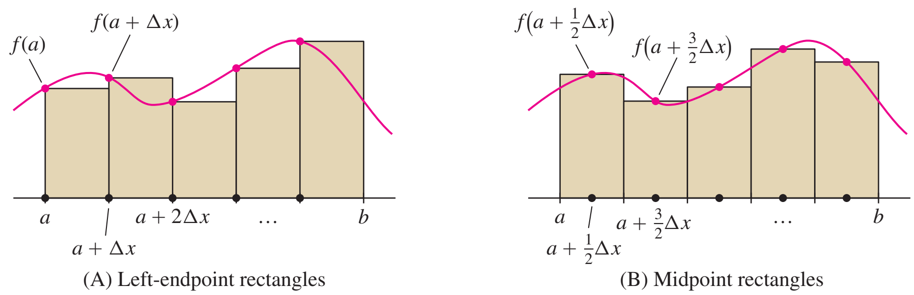
$$\frac{x_0 + x_1}{2}, \quad \frac{x_1 + x_2}{2}, \quad \ldots, \quad \frac{x_{N-1} + x_N}{2}$$

The sum of the areas of the midpoint rectangles is

$$M_N = \Delta x \left(f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{N-1} + x_N}{2}\right) \right)$$

In summation notation,

$$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$



DF **FIGURE 6**

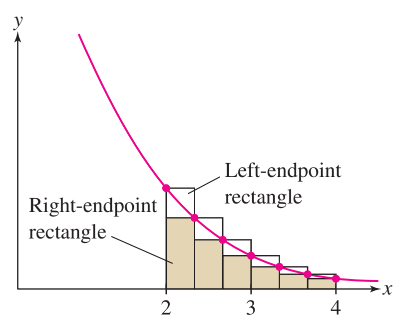


FIGURE 7 L_6 and R_6 for $f(x) = x^{-1}$ on [2, 4].

■ **EXAMPLE 2** Calculate R_6, L_6, and M_6 for $f(x) = x^{-1}$ on [2, 4].

Solution In this case, $\Delta x = (b - a)/N = (4 - 2)/6 = \frac{1}{3}$. The general term in the summation for R_6 and L_6 is

$$f(x_j) = f(a + j\Delta x) = f\left(2 + j\left(\frac{1}{3}\right)\right) = \frac{1}{2 + \frac{1}{3}j} = \frac{3}{6 + j}$$

Therefore (Figure 7),

$$R_6 = \frac{1}{3}\sum_{j=1}^{6} f(x_j) = \frac{1}{3}\sum_{j=1}^{6} \frac{3}{6 + j}$$

$$= \frac{1}{3}\left(\frac{3}{7} + \frac{3}{8} + \frac{3}{9} + \frac{3}{10} + \frac{3}{11} + \frac{3}{12}\right) \approx 0.653$$

In L_6, the sum begins at $j = 0$ and ends at $j = 5$:

$$L_6 = \frac{1}{3}\sum_{j=0}^{5} \frac{3}{6 + j} = \frac{1}{3}\left(\frac{3}{6} + \frac{3}{7} + \frac{3}{8} + \frac{3}{9} + \frac{3}{10} + \frac{3}{11}\right) \approx 0.737$$

The general term in M_6 is

$$f\left(\frac{x_j + x_{j+1}}{2}\right)$$

In this case, the midpoints are $\frac{13}{6}, \frac{15}{6}, \frac{17}{6}, \frac{19}{6}, \frac{21}{6}$ and $\frac{23}{6}$. Summing up from $j = 0$ to 5, we obtain (Figure 8)

$$M_6 = \frac{1}{3}\left(f\left(\frac{13}{6}\right) + f\left(\frac{15}{6}\right) + f\left(\frac{17}{6}\right) + f\left(\frac{19}{6}\right) + f\left(\frac{21}{6}\right) + f\left(\frac{23}{6}\right)\right)$$

$$= \frac{1}{3}\left(\frac{6}{13} + \frac{6}{15} + \frac{6}{17} + \frac{6}{19} + \frac{6}{21} + \frac{6}{23}\right) \approx 0.692$$ ■

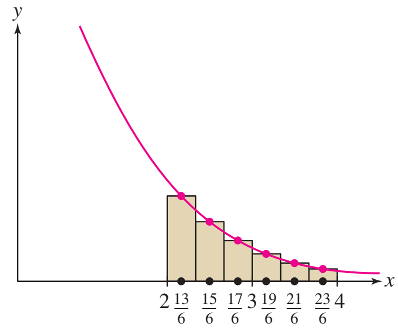


FIGURE 8 M_6 for $f(x) = x^{-1}$ on $[2, 4]$.

GRAPHICAL INSIGHT Monotonic Functions Observe in Figure 7 that the left-endpoint rectangles for $f(x) = x^{-1}$ extend above the graph and the right-endpoint rectangles lie below it. The exact area A must lie between R_6 and L_6, and so, according to the previous example, $0.65 \le A \le 0.74$. More generally, *when f is monotonic (increasing or decreasing), the exact area lies between R_N and L_N* (Figure 9):

- f increasing $\Rightarrow$ $L_N \le$ area under graph $\le R_N$
- f decreasing $\Rightarrow$ $R_N \le$ area under graph $\le L_N$

Notice that M_6 lies between R_6 and L_6. This is always the case for a monotonic function. (See Problem 93.)

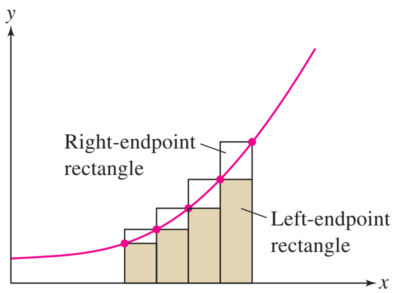


FIGURE 9 When f is increasing, the left-endpoint rectangles lie below the graph and right-endpoint rectangles lie above it.

Computing Area as the Limit of Approximations

Figure 10 shows several right-endpoint approximations. Notice that the *error* in computing the area, corresponding to the yellow region above the graph, gets smaller as the number of rectangles increases. In fact, it appears that *we can make the error as small as we please by taking the number N of rectangles large enough*. If so, it makes sense to consider the limit as $N \to \infty$, as this should give us the exact area under the curve. The next theorem guarantees that the limit exists (see Theorem 8 in Appendix D for a proof and Exercise 89 for a special case).

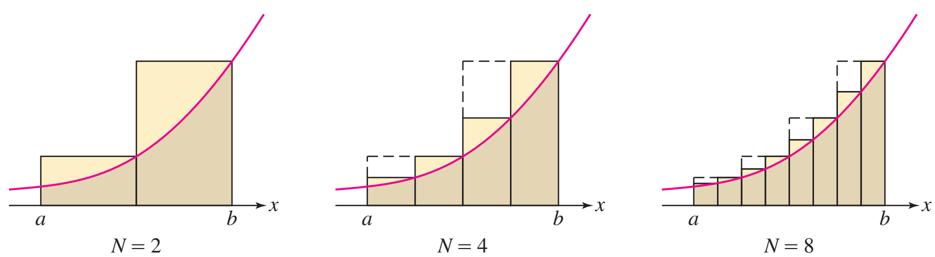


FIGURE 10 The error decreases as we use more rectangles.

THEOREM 1 If f is continuous on $[a, b]$, then the endpoint and midpoint approximations approach one and the same limit as $N \to \infty$. In other words, there is a value L such that

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = L$$

If $f(x) \ge 0$ on $[a, b]$, we define the area under the graph over $[a, b]$ to be L.

In Theorem 1, it is not assumed that $f(x) \ge 0$. If $f(x)$ takes on negative values, the limit L no longer represents area under the graph, but we can interpret it as a "signed area," discussed in the next section.

CONCEPTUAL INSIGHT In calculus, limits are used to define basic quantities that otherwise would not have a precise meaning. Theorem 1 allows us to *define* area as a limit L in much the same way that we define the slope of a tangent line as the limit of slopes of secant lines.

The next three examples illustrate Theorem 1 using formulas for **power sums**. The kth power sum is defined as the sum of the kth powers of the first N integers. We shall use the power sum formulas for $k = 1, 2, 3$.

A method for proving power sum formulas is developed in Exercises 44–47 of Section 1.3. Formulas (3)–(5) can also be verified using the method of induction.

Power Sums

$$\sum_{j=1}^{N} j = 1 + 2 + \cdots + N = \frac{N(N+1)}{2} = \frac{N^2}{2} + \frac{N}{2} \tag{3}$$

$$\sum_{j=1}^{N} j^2 = 1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \tag{4}$$

$$\sum_{j=1}^{N} j^3 = 1^3 + 2^3 + \cdots + N^3 = \frac{N^2(N+1)^2}{4} = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4} \tag{5}$$

For example, by Eq. (4),

$$\sum_{j=1}^{6} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \underbrace{\frac{6^3}{3} + \frac{6^2}{2} + \frac{6}{6}}_{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6} \text{ for } N=6} = 91$$

As a first illustration, we compute the area of a right triangle "the hard way."

■ **EXAMPLE 3** Find the area A under the graph of $f(x) = x$ over $[0, 4]$ in three ways:

(a) Using geometry **(b)** $\lim_{N\to\infty} R_N$ **(c)** $\lim_{N\to\infty} L_N$

Solution The region under the graph is a right triangle with base $b = 4$ and height $h = 4$ (Figure 11).

(a) By geometry, $A = \frac{1}{2}bh = \left(\frac{1}{2}\right)(4)(4) = 8$.

(b) We compute this area again as a limit. Since $\Delta x = (b - a)/N = 4/N$ and $f(x) = x$,

REMINDER

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j)$$

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

$$\Delta x = \frac{b-a}{N}$$

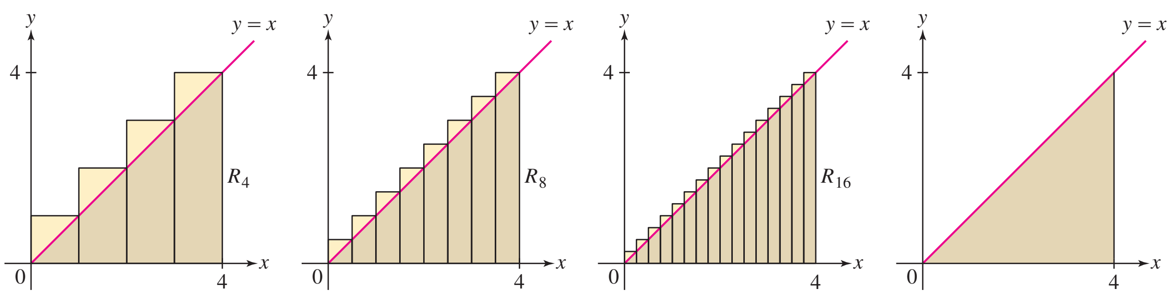
$$x_j = a + j\Delta x$$

$$f(x_j) = f(a + j\Delta x) = f\left(0 + j\left(\frac{4}{N}\right)\right) = \frac{4j}{N}$$

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{4}{N}\sum_{j=1}^{N} \frac{4j}{N} = \frac{16}{N^2}\sum_{j=1}^{N} j$$

In the last equality, we factored out $4/N$ from the sum. This is valid because $4/N$ is a constant that does not depend on j. Now use formula (3):

$$R_N = \frac{16}{N^2}\sum_{j=1}^{N} j = \frac{16}{N^2}\underbrace{\left(\frac{N(N+1)}{2}\right)}_{\text{Formula for power sum}} = \frac{8}{N^2}\left(N^2 + N\right) = 8 + \frac{8}{N}$$



DF **FIGURE 11** The right-endpoint approximations approach the area of the triangle.

The second term $8/N$ tends to zero as N approaches ∞, so

$$A = \lim_{N\to\infty} R_N = \lim_{N\to\infty} \left(8 + \frac{8}{N}\right) = 8$$

As expected, this limit yields the same value as the formula $\frac{1}{2}bh$.

(c) The left-endpoint approximation is similar, but the sum begins at $j = 0$ and ends at $j = N - 1$:

In Eq. (6), we apply the formula

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}$$

with $N - 1$ in place of N:

$$\sum_{j=1}^{N-1} j = \frac{(N-1)N}{2}$$

$$L_N = \frac{16}{N^2}\sum_{j=0}^{N-1} j = \frac{16}{N^2}\sum_{j=1}^{N-1} j = \frac{16}{N^2}\left(\frac{(N-1)N}{2}\right) = 8 - \frac{8}{N} \qquad \boxed{6}$$

Note in the second step that we replaced the sum beginning at $j = 0$ with a sum beginning at $j = 1$. This is valid because the term for $j = 0$ is zero and may be dropped. Again, we find that $A = \lim_{N\to\infty} L_N = \lim_{N\to\infty} (8 - 8/N) = 8$. ■

In the next example, we compute the area under a curved graph. Unlike the previous example, it is not possible to compute this area directly using geometry.

■ **EXAMPLE 4** Let A be the area under the graph of $f(x) = 2x^2 - x + 3$ over $[2, 4]$ (Figure 12). Compute A as the limit $\lim_{N\to\infty} R_N$.

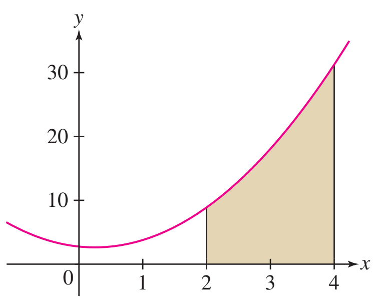


FIGURE 12 Area under the graph of $f(x) = 2x^2 - x + 3$ over $[2, 4]$.

Solution

Step 1. **Express R_N in terms of power sums.**

In this case, $\Delta x = (4 - 2)/N = 2/N$ and

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \Delta x \sum_{j=1}^{N} f(a + j\Delta x) = \frac{2}{N}\sum_{j=1}^{N} f\left(2 + \frac{2j}{N}\right)$$

Let's use algebra to simplify the general term. Since $f(x) = 2x^2 - x + 3$,

$$f\left(2 + \frac{2j}{N}\right) = 2\left(2 + \frac{2j}{N}\right)^2 - \left(2 + \frac{2j}{N}\right) + 3$$

$$= 2\left(4 + \frac{8j}{N} + \frac{4j^2}{N^2}\right) - \left(2 + \frac{2j}{N}\right) + 3 = \frac{8}{N^2}j^2 + \frac{14}{N}j + 9$$

Now we can express R_N in terms of power sums:

$$R_N = \frac{2}{N}\sum_{j=1}^{N}\left(\frac{8}{N^2}j^2 + \frac{14}{N}j + 9\right) = \frac{2}{N}\sum_{j=1}^{N}\frac{8}{N^2}j^2 + \frac{2}{N}\sum_{j=1}^{N}\frac{14}{N}j + \frac{2}{N}\sum_{j=1}^{N}9$$

$$= \frac{16}{N^3}\sum_{j=1}^{N} j^2 + \frac{28}{N^2}\sum_{j=1}^{N} j + \frac{18}{N}\sum_{j=1}^{N} 1 \qquad \boxed{7}$$

Step 2. **Use the formulas for the power sums.**

Using formulas (3) and (4) for the power sums in Eq. (7), we obtain

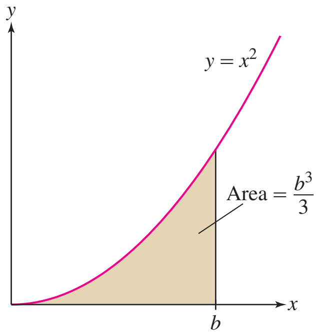
$$R_N = \frac{16}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{28}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{18}{N}(N)$$

$$= \left(\frac{16}{3} + \frac{8}{N} + \frac{8}{3N^2}\right) + \left(14 + \frac{14}{N}\right) + 18$$

$$= \frac{112}{3} + \frac{22}{N} + \frac{8}{3N^2}$$

Step 3. **Calculate the limit.**

$$A = \lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{112}{3} + \frac{22}{N} + \frac{8}{3N^2}\right) = \frac{112}{3}$$ ■



DF **FIGURE 13**

■ **EXAMPLE 5** Prove that for all $b > 0$, the area A under the graph of $f(x) = x^2$ over $[0, b]$ is equal to $b^3/3$, as indicated in Figure 13.

Solution We'll compute with R_N. We have $\Delta x = (b-0)/N = b/N$ and

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) = \frac{b}{N}\sum_{j=1}^{N}\left(0 + j\frac{b}{N}\right)^2 = \frac{b}{N}\sum_{j=1}^{N}\left(j^2\frac{b^2}{N^2}\right)$$

$$= \frac{b^3}{N^3}\sum_{j=1}^{N} j^2$$

By the formula for the power sum recalled in the margin,

REMINDER By Eq. (4),

$$\sum_{j=1}^{N} j^2 = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}$$

$$R_N = \frac{b^3}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) = \frac{b^3}{3} + \frac{b^3}{2N} + \frac{b^3}{6N^2}$$

$$A = \lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{b^3}{3} + \frac{b^3}{2N} + \frac{b^3}{6N^2}\right) = \frac{b^3}{3}$$ ■

The area under the graph of any polynomial can be calculated using power sum formulas as in the examples above. For other functions, the limit defining the area may be difficult or impossible to evaluate directly. Consider $f(x) = \sin x$ on the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$. In this case (Figure 14), $\Delta x = (3\pi/4 - \pi/4)/N = \pi/(2N)$ and the area A is

$$A = \lim_{N\to\infty} R_N = \lim_{N\to\infty} \Delta x \sum_{j=1}^{N} f(a + j\Delta x) = \lim_{N\to\infty} \frac{\pi}{2N}\sum_{j=1}^{N}\sin\left(\frac{\pi}{4} + \frac{\pi j}{2N}\right)$$

With some work, we can show that the limit is equal to $A = \sqrt{2}$. However, in Section 5.4, we will see that it is much easier to apply the Fundamental Theorem of Calculus, which reduces area computations to the problem of finding antiderivatives.

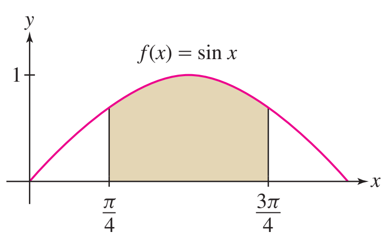


FIGURE 14 The area of this region is more difficult to compute as a limit of endpoint approximations.

HISTORICAL PERSPECTIVE

(© Bettmann/Corbis)

Jacob Bernoulli
(1654–1705)

We used the formulas for the kth power sums for $k = 1, 2, 3$. Do similar formulas exist for all powers k? This problem was studied in the seventeenth century and eventually solved around 1690 by the great Swiss mathematician Jacob Bernoulli. Of this discovery, he wrote

> With the help of [these formulas] it took me less than half of a quarter of an hour to find that the 10th powers of the first 1000 numbers being added together will yield the sum
>
> 91409924241424243424241924242500

Bernoulli's formula has the general form

$$\sum_{j=1}^{n} j^k = \frac{1}{k+1}n^{k+1} + \frac{1}{2}n^k + \frac{k}{12}n^{k-1} + \cdots$$

The dots indicate terms involving smaller powers of n whose coefficients are expressed in terms of the so-called Bernoulli numbers. For example,

$$\sum_{j=1}^{n} j^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

These formulas are available on most computer algebra systems.

5.1 SUMMARY

Power Sums

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2} = \frac{N^2}{2} + \frac{N}{2}$$

$$\sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}$$

$$\sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4} = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}$$

- Approximations to the area under the graph of f over the interval $[a, b]$ $\left(\Delta x = \dfrac{b-a}{N}, x_j = a + j\Delta x\right)$:

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \Delta x\big(f(x_1) + f(x_2) + \cdots + f(x_N)\big)$$

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \Delta x\big(f(x_0) + f(x_1) + \cdots + f(x_{N-1})\big)$$

$$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$

$$= \Delta x\left(f\left(\frac{x_0 + x_1}{2}\right) + \cdots + f\left(\frac{x_{N-1} + x_N}{2}\right)\right)$$

- If f is continuous on $[a, b]$, then the endpoint and midpoint approximations approach one and the same limit L:

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = L$$

- If $f(x) \ge 0$ on $[a, b]$, we take L as the definition of the area under the graph of $y = f(x)$ over $[a, b]$.

5.1 EXERCISES

Preliminary Questions

1. What are the right and left endpoints if [2, 5] is divided into six subintervals?

2. The interval [1, 5] is divided into eight subintervals.

(a) What is the left endpoint of the last subinterval?

(b) What are the right endpoints of the first two subintervals?

3. Which of the following pairs of sums are *not* equal?

(a) $\sum_{i=1}^{4} i, \quad \sum_{\ell=1}^{4} \ell$ **(b)** $\sum_{j=1}^{4} j^2, \quad \sum_{k=2}^{5} k^2$

(c) $\sum_{j=1}^{4} j, \quad \sum_{i=2}^{5} (i-1)$ **(d)** $\sum_{i=1}^{4} i(i+1), \quad \sum_{j=2}^{5} (j-1)j$

4. Explain: $\sum_{j=1}^{100} j = \sum_{j=0}^{100} j$ but $\sum_{j=1}^{100} 1$ is not equal to $\sum_{j=0}^{100} 1$.

5. Explain why $L_{100} \ge R_{100}$ for $f(x) = x^{-2}$ on [3, 7].

Exercises

1. Figure 15 shows the velocity of an object over a 3-minute (min) interval. Determine the distance traveled over the intervals [0, 3] and [1, 2.5] (remember to convert from kilometers per hour to kilometers per minute).

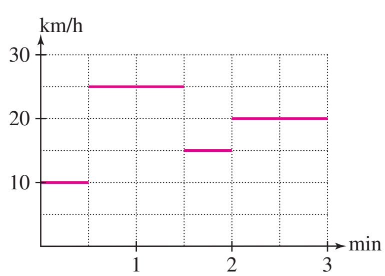


FIGURE 15

2. An ostrich (Figure 16) runs with velocity 20 km/h for 2 minutes (min), 12 km/h for 3 min, and 40 km/h for another minute. Compute the total distance traveled and indicate with a graph how this quantity can be interpreted as an area.

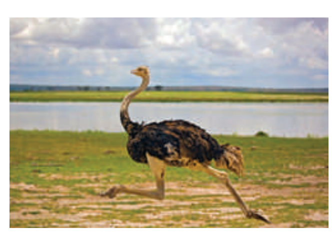


FIGURE 16 Ostriches can reach speeds as high as 70 km/h. (© *Daryl Balfour/Gallo Images/Alamy*)

3. A rainstorm hit Portland, Maine, in October 1996, resulting in record rainfall. The rainfall rate $R(t)$ on October 21 is recorded, in centimeters per hour, in the following table, where t is the number of hours since midnight. Compute the total rainfall during this 24-hour period and indicate on a graph how this quantity can be interpreted as an area.

t (h)	0–2	2–4	4–9	9–12	12–20	20–24
$R(t)$ (cm)	0.5	0.3	1.0	2.5	1.5	0.6

4. The velocity of an object is $v(t) = 12t$ m/s. Use Eq. (2) and geometry to find the distance traveled over the time intervals $[0, 2]$ and $[2, 5]$.

5. Compute R_5 and L_5 over $[0, 1]$ using the following values:

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	50	48	46	44	42	40

6. Compute R_6, L_6, and M_3 to estimate the distance traveled over $[0, 3]$ if the velocity at half-second intervals is as follows:

t (s)	0	0.5	1	1.5	2	2.5	3
v (m/s)	0	12	18	25	20	14	20

7. Let $f(x) = 2x + 3$.

(a) Compute R_6 and L_6 over $[0, 3]$.

(b) Use geometry to find the exact area A and compute the errors $|A - R_6|$ and $|A - L_6|$ in the approximations.

8. Repeat Exercise 7 for $f(x) = 20 - 3x$ over $[2, 4]$.

9. Calculate R_3 and L_3 for $f(x) = x^2 - x + 4$ over $[1, 4]$. Then sketch the graph of f and the rectangles that make up each approximation. Is the area under the graph larger or smaller than R_3? Is it larger or smaller than L_3?

10. Let $f(x) = \sqrt{x^2 + 1}$ and $\Delta x = \frac{1}{3}$. Sketch the graph of f and draw the right-endpoint rectangles whose area is represented by the sum $\sum_{i=1}^{6} f(1 + i\Delta x)\Delta x$.

11. Estimate R_3, M_3, and L_6 over $[0, 1.5]$ for the function in Figure 17.

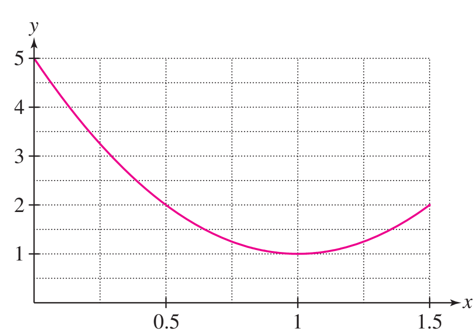


FIGURE 17

12. Calculate the area of the shaded rectangles in Figure 18. Which approximation do these rectangles represent?

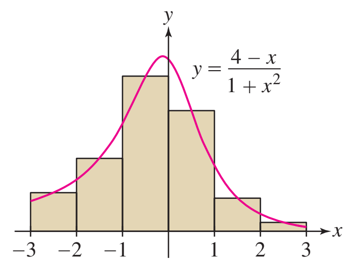


FIGURE 18

13. Let $f(x) = x^2$.

(a) Sketch the function over the interval $[0, 2]$ and the rectangles corresponding to L_4. Calculate the area contained within them.

(b) Sketch the function over the interval $[0, 2]$ again but with the rectangles corresponding to R_4. Calculate the area contained within them.

(c) Make a conclusion about the area under the curve $f(x) = x^2$ over the interval $[0, 2]$.

14. Let $f(x) = \sqrt{x}$.

(a) Sketch the function over the interval $[0, 4]$ and the rectangles corresponding to L_4. Calculate the area contained within them.

(b) Sketch the function over the interval $[0, 4]$ again but with the rectangles corresponding to R_4. Calculate the area contained within them.

(c) Make a conclusion about the area under the curve $f(x) = \sqrt{x}$ over the interval $[0, 4]$.

In Exercises 15–22, calculate the approximation for the given function and interval.

15. R_3, $\quad f(x) = 7 - x$, $\quad [3, 5]$

16. L_6, $\quad f(x) = \sqrt{6x + 2}$, $\quad [1, 3]$

17. M_6, $\quad f(x) = 4x + 3$, $\quad [5, 8]$

18. R_5, $\quad f(x) = x^2 + x$, $\quad [-1, 1]$

19. M_5, $\quad f(x) = \ln x$, $\quad [1, 3]$

20. M_4, $\quad f(x) = \sqrt{x}$, $\quad [3, 5]$

21. L_4, $\quad f(x) = \cos^2 x$, $\quad \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

22. L_6, $\quad f(x) = x^2 + 3|x|$, $\quad [-2, 1]$

In Exercises 23–28, write the sum in summation notation.

23. $4^7 + 5^7 + 6^7 + 7^7 + 8^7$

24. $(2^2 + 2) + (3^2 + 3) + (4^2 + 4) + (5^2 + 5)$

25. $(2^2 + 2) + (2^3 + 2) + (2^4 + 2) + (2^5 + 2)$

26. $\sqrt{1 + 1^3} + \sqrt{2 + 2^3} + \cdots + \sqrt{n + n^3}$

27. $\dfrac{1}{2 \cdot 3} + \dfrac{2}{3 \cdot 4} + \cdots + \dfrac{n}{(n+1)(n+2)}$

28. $e^{\pi} + e^{\pi/2} + e^{\pi/3} + \cdots + e^{\pi/n}$

29. Calculate the sums:

(a) $\sum_{i=1}^{5} 9$ **(b)** $\sum_{i=0}^{5} 4$ **(c)** $\sum_{k=2}^{4} k^3$

30. Calculate the sums:

(a) $\sum_{j=3}^{4} \sin\left(j\frac{\pi}{2}\right)$ **(b)** $\sum_{k=3}^{5} \frac{1}{k-1}$ **(c)** $\sum_{j=0}^{2} 3^{j-1}$

31. Let $b_1 = 4$, $b_2 = 1$, $b_3 = 2$, and $b_4 = -4$. Calculate:

(a) $\sum_{i=2}^{4} b_i$ **(b)** $\sum_{j=1}^{2} (2^{b_j} - b_j)$ **(c)** $\sum_{k=1}^{3} k b_k$

32. Assume that $a_1 = -5$, $\sum_{i=1}^{10} a_i = 20$, and $\sum_{i=1}^{10} b_i = 7$. Calculate:

(a) $\sum_{i=1}^{10} (4a_i + 3)$ **(b)** $\sum_{i=2}^{10} a_i$ **(c)** $\sum_{i=1}^{10} (2a_i - 3b_i)$

33. Calculate $\sum_{j=101}^{200} j$. *Hint:* Write as a difference of two sums and use formula (3).

34. Calculate $\sum_{j=1}^{30} (2j+1)^2$. *Hint:* Expand and use formulas (3)–(4).

In Exercises 35–42, use linearity and formulas (3)–(5) to rewrite and evaluate the sums.

35. $\sum_{j=1}^{20} 8j^3$

36. $\sum_{k=1}^{30} (4k - 3)$

37. $\sum_{n=51}^{150} n^2$

38. $\sum_{k=101}^{200} k^3$

39. $\sum_{j=0}^{50} j(j-1)$

40. $\sum_{j=2}^{30} \left(6j + \frac{4j^2}{3}\right)$

41. $\sum_{m=1}^{30} (4-m)^3$

42. $\sum_{m=1}^{20} \left(5 + \frac{3m}{2}\right)^2$

In Exercises 43–46, use formulas (3)–(5) to evaluate the limit.

43. $\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N^2}$

44. $\lim_{N\to\infty} \sum_{j=1}^{N} \frac{j^3}{N^4}$

45. $\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3}$

46. $\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{i^3}{N^4} - \frac{20}{N}\right)$

In Exercises 47–52, calculate the limit for the given function and interval. Verify your answer by using geometry.

47. $\lim_{N\to\infty} R_N$, $f(x) = 9x$, $[0, 2]$

48. $\lim_{N\to\infty} R_N$, $f(x) = 3x + 6$, $[1, 4]$

49. $\lim_{N\to\infty} L_N$, $f(x) = \frac{1}{2}x + 2$, $[0, 4]$

50. $\lim_{N\to\infty} L_N$, $f(x) = 4x - 2$, $[1, 3]$

51. $\lim_{N\to\infty} M_N$, $f(x) = x$, $[0, 2]$

52. $\lim_{N\to\infty} M_N$, $f(x) = 12 - 4x$, $[2, 6]$

53. Show, for $f(x) = 3x^2 + 4x$ over $[0, 2]$, that

$$R_N = \frac{2}{N} \sum_{j=1}^{N} \left(\frac{12j^2}{N^2} + \frac{8j}{N}\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

54. Show, for $f(x) = 3x^3 - x^2$ over $[1, 5]$, that

$$R_N = \frac{4}{N} \sum_{j=1}^{N} \left(\frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

In Exercises 55–62, find a formula for R_N and compute the area under the graph as a limit.

55. $f(x) = x^2$, $[0, 1]$

56. $f(x) = x^2$, $[-1, 5]$

57. $f(x) = 6x^2 - 4$, $[2, 5]$

58. $f(x) = x^2 + 7x$, $[6, 11]$

59. $f(x) = x^3 - x$, $[0, 2]$

60. $f(x) = 2x^3 + x^2$, $[-2, 2]$

61. $f(x) = 2x + 1$, $[a, b]$ (a, b constants with $a < b$)

62. $f(x) = x^2$, $[a, b]$ (a, b constants with $a < b$)

In Exercises 63–66, describe the area represented by the limits.

63. $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\frac{j}{N}\right)^4$

64. $\lim_{N\to\infty} \frac{3}{N} \sum_{j=1}^{N} \left(2 + \frac{3j}{N}\right)^4$

65. $\lim_{N\to\infty} \frac{5}{N} \sum_{j=0}^{N-1} e^{-2+5j/N}$

66. $\lim_{N\to\infty} \frac{\pi}{2N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} - \frac{\pi}{4N} + \frac{j\pi}{2N}\right)$

In Exercises 67–72, express the area under the graph as a limit using the approximation indicated (in summation notation), but do not evaluate.

67. R_N, $f(x) = \sin x$ over $[0, \pi]$

68. R_N, $f(x) = x^{-1}$ over $[1, 7]$

69. L_N, $f(x) = \sqrt{2x+1}$ over $[7, 11]$

70. L_N, $f(x) = \cos x$ over $\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$

71. M_N, $f(x) = \tan x$ over $\left[\frac{1}{2}, 1\right]$

72. M_N, $f(x) = x^{-2}$ over $[3, 5]$

73. Evaluate $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left(\frac{j}{N}\right)^2}$ by interpreting it as the area of part of a familiar geometric figure.

In Exercises 74–76, let $f(x) = x^2$ and let R_N, L_N, and M_N be the approximations for the interval $[0, 1]$.

74. Show that $R_N = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$. Interpret the quantity $\frac{1}{2N} + \frac{1}{6N^2}$ as the area of a region.

75. Show that

$$L_N = \frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2}, \qquad M_N = \frac{1}{3} - \frac{1}{12N^2}$$

Then rank the three approximations R_N, L_N, and M_N in order of increasing accuracy (use Exercise 74).

76. For each of R_N, L_N, and M_N, find the smallest integer N for which the error is less than 0.001.

In Exercises 77–82, use the Graphical Insight on page 264 to obtain bounds on the area.

77. Let A be the area under $f(x) = \sqrt{x}$ over [0, 1]. Prove that $0.51 \le A \le 0.77$ by computing R_4 and L_4. Explain your reasoning.

78. Use R_5 and L_5 to show that the area A under $y = x^{-2}$ over [10, 13] satisfies $0.0218 \le A \le 0.0244$.

79. Use R_4 and L_4 to show that the area A under the graph of $y = \sin x$ over $\left[0, \frac{\pi}{2}\right]$ satisfies $0.79 \le A \le 1.19$.

80. Show that the area A under $f(x) = x^{-1}$ over [1, 8] satisfies

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \le A \le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

81. CAS Show that the area A under $y = x^{1/4}$ over [0, 1] satisfies $L_N \le A \le R_N$ for all N. Use a computer algebra system to calculate L_N and R_N for $N = 100$ and 200, and determine A to two decimal places.

82. CAS Show that the area A under $y = 4/(x^2 + 1)$ over [0, 1] satisfies $R_N \le A \le L_N$ for all N. Determine A to at least three decimal places using a computer algebra system. Can you guess the exact value of A?

83. In this exercise, we evaluate the area A under the graph of $y = e^x$ over [0, 1] [Figure 19(A)] using the formula for a geometric sum (valid for $r \ne 1$):

$$1 + r + r^2 + \cdots + r^{N-1} = \sum_{j=0}^{N-1} r^j = \frac{r^N - 1}{r - 1} \qquad \boxed{8}$$

(a) Show that $L_N = \dfrac{1}{N} \displaystyle\sum_{j=0}^{N-1} e^{j/N}$.

(b) Apply Eq. (8) with $r = e^{1/N}$ to prove $L_N = \dfrac{e - 1}{N(e^{1/N} - 1)}$.

(c) Compute $A = \lim_{N\to\infty} L_N$ using L'Hôpital's Rule.

84. Use the result of Exercise 83 to show that the area B under the graph of $f(x) = \ln x$ over $[1, e]$ is equal to 1. *Hint:* Relate B in Figure 19(B) to the area A computed in Exercise 83.

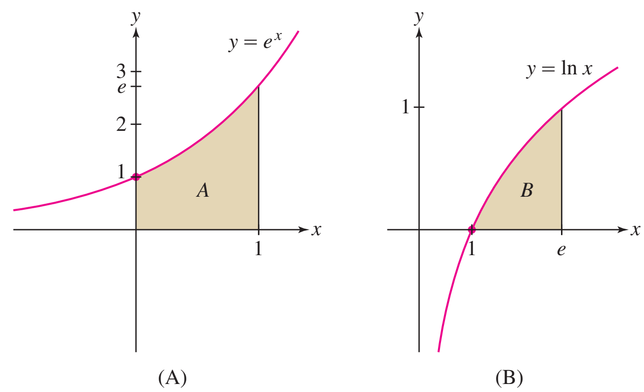


FIGURE 19

Further Insights and Challenges

85. Although the accuracy of R_N generally improves as N increases, this need not be true for small values of N. Draw the graph of a positive continuous function f on an interval such that R_1 is closer than R_2 to the exact area under the graph. Can such a function be monotonic?

86. Draw the graph of a positive continuous function on an interval such that R_2 and L_2 are both smaller than the exact area under the graph. Can such a function be monotonic?

87. Explain graphically: *The endpoint approximations are less accurate when $f'(x)$ is large.*

88. Prove that for any function f on $[a, b]$,

$$R_N - L_N = \frac{b - a}{N}(f(b) - f(a)) \qquad \boxed{9}$$

89. In this exercise, we prove that $\lim_{N\to\infty} R_N$ and $\lim_{N\to\infty} L_N$ exist and are equal if f is increasing [the case of f decreasing is similar]. We use the concept of a least upper bound discussed in Appendix B.

(a) Explain with a graph why $L_N \le R_M$ for all $N, M \ge 1$.

(b) By (a), the sequence $\{L_N\}$ is bounded, so it has a least upper bound L. By definition, L is the smallest number such that $L_N \le L$ for all N. Show that $L \le R_M$ for all M.

(c) According to (b), $L_N \le L \le R_N$ for all N. Use Eq. (9) to show that $\lim_{N\to\infty} L_N = L$ and $\lim_{N\to\infty} R_N = L$.

90. Use Eq. (9) to show that if f is positive and monotonic, then the area A under its graph over $[a, b]$ satisfies

$$|R_N - A| \le \frac{b - a}{N}|f(b) - f(a)| \qquad \boxed{10}$$

In Exercises 91–92, use Eq. (10) to find a value of N such that $|R_N - A| < 10^{-4}$ for the given function and interval.

91. $f(x) = \sqrt{x}$, [1, 4]

92. $f(x) = \sqrt{9 - x^2}$, [0, 3]

93. Prove that if f is positive and monotonic, then M_N lies between R_N and L_N and is closer to the actual area under the graph than both R_N and L_N. *Hint:* In the case that f is increasing, Figure 20 shows that the part of the error in R_N due to the ith rectangle is the sum of the areas $A + B + D$, and for M_N it is $|B - E|$. On the other hand, $A \ge E$.

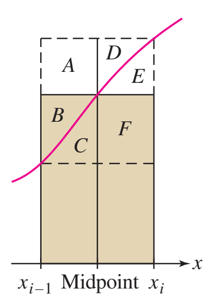


FIGURE 20

5.2 The Definite Integral

In the previous section, we saw that if f is continuous on an interval $[a, b]$, then the endpoint and midpoint approximations approach a common limit L as $N \to \infty$:

$$L = \lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N \tag{1}$$

When $f(x) \geq 0$, L is the area under the graph of f. In a moment, we will state formally that L is the *definite integral* of f over $[a, b]$. Before doing so, we introduce more general approximations called **Riemann sums**.

Recall that R_N, L_N, and M_N use rectangles of equal width Δx, whose heights are the values of $f(x)$ at the endpoints or midpoints of the subintervals. In Riemann sum approximations, we relax these requirements: The rectangles need not have equal width, and the height may be *any* value of $f(x)$ within the subinterval.

To specify a Riemann sum, we choose a partition and a set of sample points:

- **Partition** P of size N: a choice of points that divides $[a, b]$ into N subintervals.

$$P : a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

- **Sample points** $C = \{c_1, \ldots, c_N\}$: c_i belongs to the subinterval $[x_{i-1}, x_i]$ for all $i = 1, \ldots, N$.

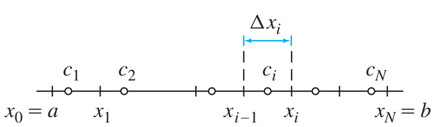


FIGURE 1 Partition of size N and set of sample points.

See Figures 1 and 2(A). The length of the ith subinterval $[x_{i-1}, x_i]$ is

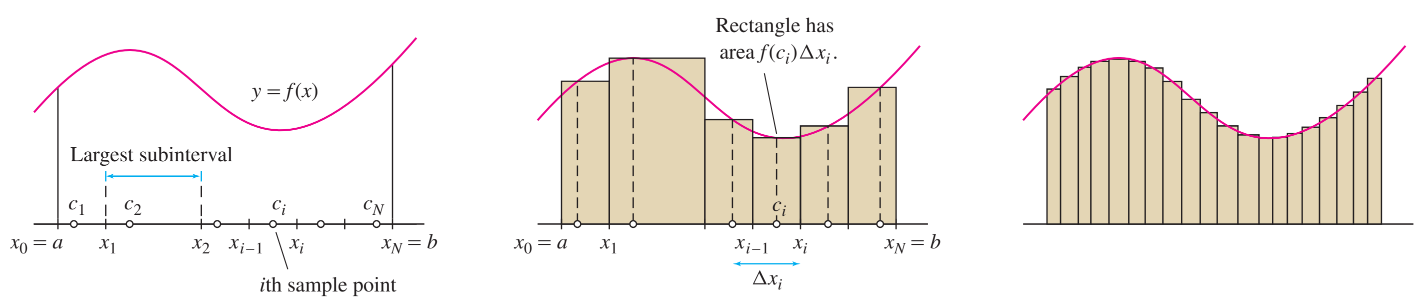
$$\Delta x_i = x_i - x_{i-1}$$

The **norm** of P, denoted $\|P\|$, is the maximum of the lengths Δx_i.

Given P and C, we construct the rectangle of height $f(c_i)$ and base Δx_i over each subinterval $[x_{i-1}, x_i]$, as in Figure 2(B). This rectangle has area $f(c_i)\Delta x_i$ if $f(c_i) \geq 0$. If $f(c_i) < 0$, the rectangle extends below the x-axis, and $f(c_i)\Delta x_i$ is the negative of its area. The Riemann sum is the sum

Keep in mind that R_N, L_N, and M_N are particular examples of Riemann sums in which $\Delta x_i = (b - a)/N$ for all i, and the sample points c_i are either endpoints or midpoints.

$$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_N)\Delta x_N \tag{2}$$



(A) Partition of $[a, b]$ into subintervals.

(B) Construct rectangle above each subinterval of height $f(c_i)$.

(C) Rectangles corresponding to a Riemann sum with $\|P\|$ small (a large number of rectangles).

FIGURE 2 Construction of $R(f, P, C)$. In this case, Δx_2 is the norm of the partition.

■ **EXAMPLE 1** Calculate the Riemann sum $R(f, P, C)$, where $f(x) = 8 + 12 \sin x - 4x$ on $[0, 4]$,

$$P : x_0 = 0 < x_1 = 1 < x_2 = 1.8 < x_3 = 2.9 < x_4 = 4$$

$$C = \{0.4, 1.2, 2, 3.5\}$$

What is the norm $\|P\|$?

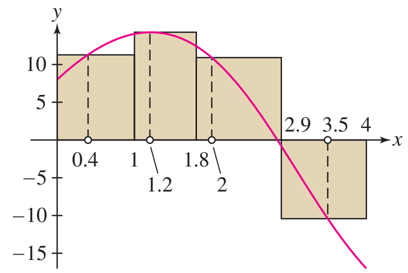


FIGURE 3 Rectangles defined by a Riemann sum for $f(x) = 8 + 12\sin x - 4x$.

Solution The widths of the subintervals in the partition (Figure 3) are

$$\Delta x_1 = x_1 - x_0 = 1 - 0 = 1, \qquad \Delta x_2 = x_2 - x_1 = 1.8 - 1 = 0.8$$

$$\Delta x_3 = x_3 - x_2 = 2.9 - 1.8 = 1.1, \qquad \Delta x_4 = x_4 - x_3 = 4 - 2.9 = 1.1$$

The norm of the partition is $\|P\| = 1.1$ since the two longest subintervals have width 1.1. Using a calculator, we obtain

$$R(f, P, C) = f(0.4)\Delta x_1 + f(1.2)\Delta x_2 + f(2)\Delta x_3 + f(3.5)\Delta x_4$$

$$\approx 11.07(1) + 14.38(0.8) + 10.91(1.1) - 10.2(1.1) \approx 23.35$$ ■

Note in Figure 2(C) that as the norm $\|P\|$ tends to zero (meaning that the rectangles get thinner), the number of rectangles N tends to ∞ and they approximate the area under the graph more closely. This leads to the following definition: f is **integrable** over $[a, b]$ if *all* of the Riemann sums (not just the endpoint and midpoint approximations) approach one and the same limit L as $\|P\|$ tends to zero. Formally, we write

$$L = \lim_{\|P\|\to 0} R(f, P, C) = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i \qquad \boxed{3}$$

if $|R(f, P, C) - L|$ gets arbitrarily small as the norm $\|P\|$ tends to zero, no matter how we choose the partition and sample points. The limit L is called the **definite integral** of f over $[a, b]$.

The notation $\int f(x)\,dx$ was introduced by Leibniz in 1686. The symbol $\int$ is an elongated S standing for "summation." The differential dx corresponds to the length Δx_i along the x-axis.

DEFINITION Definite Integral The definite integral of f over $[a, b]$, denoted by the integral sign, is the limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f, P, C) = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

When this limit exists, we say that f is integrable over $[a, b]$.

One of the greatest mathematicians of the nineteenth century and perhaps second only to his teacher C. F. Gauss, Riemann transformed the fields of geometry, analysis, and number theory. Albert Einstein based his General Theory of Relativity *on Riemann's geometry. The "Riemann Hypothesis" dealing with prime numbers is one of the great unsolved problems in present-day mathematics. The Clay Foundation has offered a $1 million prize for its solution (http://claymath.org/millenium-problems/riemann-hypothesis).*

The definite integral is often called, more simply, the *integral* of f over $[a, b]$. The process of computing integrals is called **integration** and $f(x)$ is called the **integrand**. The endpoints a and b of $[a, b]$ are called the **limits of integration**. Finally, we remark that any variable may be used as a variable of integration (this is a "dummy" variable). Thus, the following three integrals all denote the same quantity:

$$\int_a^b \sin x\,dx, \qquad \int_a^b \sin t\,dt, \qquad \int_a^b \sin u\,du$$

CONCEPTUAL INSIGHT Keep in mind that a Riemann sum $R(f, P, C)$ is nothing more than an approximation to area based on rectangles, and that $\int_a^b f(x)\,dx$ is the number we obtain in the limit as we take thinner and thinner rectangles.

However, general Riemann sums (with arbitrary partitions and sample points) are rarely used for computations. In practice, we use particular approximations such as M_N, or the Fundamental Theorem of Calculus, as we'll learn shortly. If so, why bother introducing Riemann sums? The answer is that Riemann sums play a theoretical rather than a computational role. They are useful in proofs and for dealing rigorously with certain discontinuous functions. In later sections, Riemann sums are used to show that volumes and other quantities can be expressed as definite integrals.

Georg Friedrich Riemann (1826–1866)

The next theorem assures us that continuous functions (and even functions with finitely many jump discontinuities) are integrable (see Appendix D for a proof). In practice, we rely on this theorem rather than attempting to prove directly that a given function is integrable.

THEOREM 1 If f is continuous on $[a, b]$, or if f is continuous with at most finitely many jump discontinuities, then f is integrable over $[a, b]$.

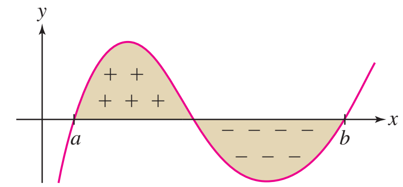
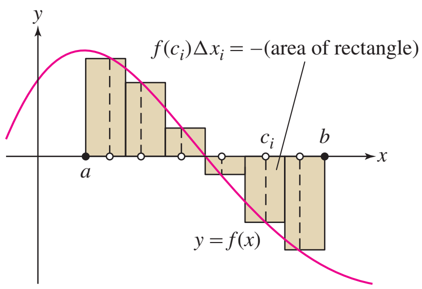


FIGURE 4 Signed area is the area above the x-axis minus the area below the x-axis.



DF FIGURE 5

Interpretation of the Definite Integral as Signed Area

When $f(x) \geq 0$, the definite integral defines the area under the graph. To interpret the integral when $f(x)$ takes on both positive and negative values, we define the notion of **signed area**, where regions below the x-axis are considered to have "negative area" (Figure 4); that is,

Signed area of a region = (area above x-axis) − (area below x-axis)

Now observe that a Riemann sum is equal to the signed area of the corresponding rectangles:

$$R(f, C, P) = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_N)\Delta x_N$$

Indeed, if $f(c_i) < 0$, then the corresponding rectangle lies below the x-axis and has signed area $f(c_i)\Delta x_i$ (Figure 5). The limit of the Riemann sums is the signed area of the region between the graph and the x-axis:

$$\int_a^b f(x)\,dx = \text{signed area of region between the graph and } x\text{-axis over } [a, b]$$

■ **EXAMPLE 2** **Signed Area** Calculate

$$\int_0^5 (3 - x)\,dx \qquad \text{and} \qquad \int_0^5 |3 - x|\,dx$$

Solution The region between $y = 3 - x$ and the x-axis consists of two triangles of areas $\frac{9}{2}$ and 2 [Figure 6(A)]. However, the second triangle lies below the x-axis, so it has signed area −2. In the graph of $y = |3 - x|$, both triangles lie above the x-axis [Figure 6(B)]. Therefore,

$$\int_0^5 (3 - x)\,dx = \frac{9}{2} - 2 = \frac{5}{2} \qquad \int_0^5 |3 - x|\,dx = \frac{9}{2} + 2 = \frac{13}{2}$$ ■

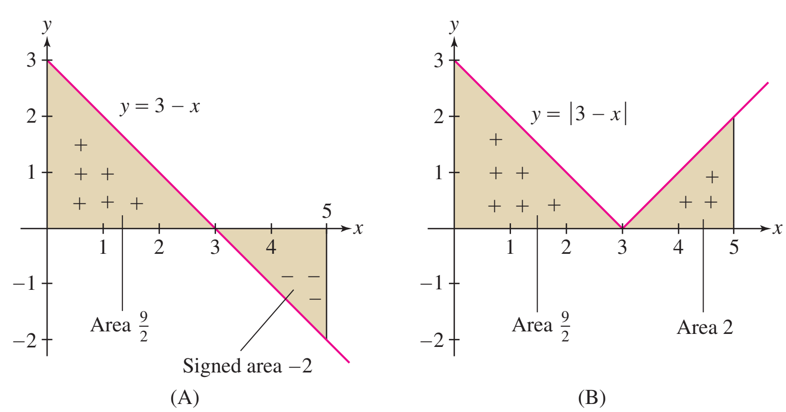


FIGURE 6

Properties of the Definite Integral

In the rest of this section, we discuss some basic properties of definite integrals. First, we note that the integral of a constant function $f(x) = C$ over $[a, b]$ is the signed area $C(b - a)$ of a rectangle (as we can see from Figure 7).

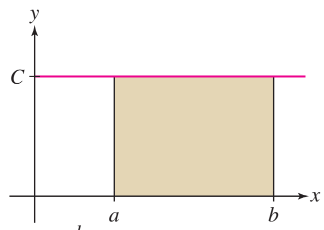


FIGURE 7 $\int_a^b C\,dx = C(b-a)$.

THEOREM 2 Integral of a Constant For any constant C,

$$\int_a^b C\,dx = C(b-a) \tag{4}$$

Next, we state the linearity properties of the definite integral.

THEOREM 3 Linearity of the Definite Integral If f and g are integrable over $[a,b]$, then $f+g$ and Cf are integrable (for any constant C), and

- $\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$

Proof These properties follow from the corresponding linearity properties of sums and limits. For example, Riemann sums are additive:

$$R(f+g,P,C) = \sum_{i=1}^{N}\big(f(c_i)+g(c_i)\big)\Delta x_i = \sum_{i=1}^{N} f(c_i)\Delta x_i + \sum_{i=1}^{N} g(c_i)\Delta x_i$$
$$= R(f,P,C) + R(g,P,C)$$

By the additivity of limits,

$$\int_a^b (f(x)+g(x))\,dx = \lim_{||P||\to 0} R(f+g,P,C)$$
$$= \lim_{||P||\to 0} R(f,P,C) + \lim_{||P||\to 0} R(g,P,C)$$
$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The second property is proved similarly. ■

■ **EXAMPLE 3** Calculate $\int_0^3 (2x^2-5)\,dx$ using the formula

$$\int_0^b x^2\,dx = \frac{b^3}{3} \tag{5}$$

Eq. (5) was verified in Example 5 of Section 5.1.

Solution

$$\int_0^3 (2x^2-5)\,dx = 2\int_0^3 x^2\,dx + \int_0^3 (-5)\,dx \qquad \text{(linearity)}$$
$$= 2\left(\frac{3^3}{3}\right) - 5(3-0) = 3 \qquad \text{[Eqs. (5) and (4)]}$$ ■

So far we have used the notation $\int_a^b f(x)\,dx$ with the understanding that $a<b$. It is convenient to define the definite integral for arbitrary a and b.

According to Eq. (6), ***the integral changes sign when the limits of integration are reversed****. Since we are free to define symbols as we please, why have we chosen to put the minus sign on the right side of Eq. (6)? Because it is only with this definition that the Fundamental Theorem of Calculus holds true.*

DEFINITION Reversing the Limits of Integration For $a<b$, we set

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{6}$$

For example, by Eq. (5),

$$\int_5^0 x^2\,dx = -\int_0^5 x^2\,dx = -\frac{5^3}{3} = -\frac{125}{3}$$

When $a = b$, the interval $[a, b] = [a, a]$ has length zero and we define the definite integral to be zero:

$$\int_a^a f(x)\,dx = 0$$

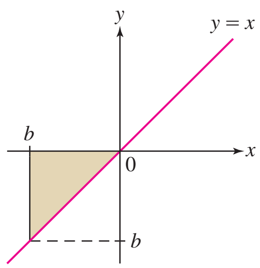


FIGURE 8 Here, $b < 0$ and the signed area is $-\frac{1}{2}b^2$.

■ **EXAMPLE 4** Prove that, for all b (positive or negative),

$$\int_0^b x\,dx = \frac{1}{2}b^2 \qquad (7)$$

Solution If $b > 0$, $\int_0^b x\,dx$ is the area $\frac{1}{2}b^2$ of a triangle of base b and height b. If $b < 0$, $\int_b^0 x\,dx$ is the signed area $-\frac{1}{2}b^2$ of the triangle in Figure 8, and Eq. (7) follows from the rule for reversing limits of integration:

$$\int_0^b x\,dx = -\int_b^0 x\,dx = -\left(-\frac{1}{2}b^2\right) = \frac{1}{2}b^2$$ ■

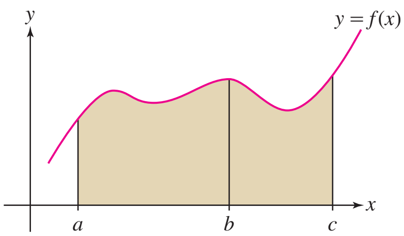


FIGURE 9 The area over $[a, c]$ is the *sum* of the areas over $[a, b]$ and $[b, c]$.

Definite integrals satisfy an important additivity property: If f is integrable and $a \le b \le c$ as in Figure 9, then the integral from a to c is equal to the integral from a to b *plus* the integral from b to c. We state this in the next theorem (a formal proof can be given using Riemann sums).

THEOREM 4 Additivity for Adjacent Intervals Let $a \le b \le c$, and assume that f is integrable. Then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

This theorem remains true as stated even if the condition $a \le b \le c$ is not satisfied (Exercise 88).

■ **EXAMPLE 5** Calculate $\int_4^7 x^2\,dx$.

Solution Before we can apply the formula $\int_0^b x^2 dx = b^3/3$ from Example 3, we must use the additivity property for adjacent intervals to write

$$\int_0^4 x^2\,dx + \int_4^7 x^2\,dx = \int_0^7 x^2\,dx$$

Now we can compute our integral as a difference:

$$\int_4^7 x^2\,dx = \int_0^7 x^2\,dx - \int_0^4 x^2\,dx = \left(\frac{1}{3}\right)7^3 - \left(\frac{1}{3}\right)4^3 = 93$$ ■

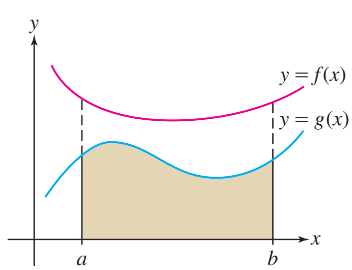


FIGURE 10 The integral of f is larger than the integral of g.

Another basic property of the definite integral is that larger functions have larger integrals (Figure 10).

THEOREM 5 Comparison Theorem If f and g are integrable and $g(x) \le f(x)$ for x in $[a, b]$, then

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$$

Proof If $g(x) \le f(x)$, then for any partition and choice of sample points, we have $g(c_i)\Delta x_i \le f(c_i)\Delta x_i$ for all i. Therefore, the Riemann sums satisfy

$$\sum_{i=1}^{N} g(c_i)\Delta x_i \le \sum_{i=1}^{N} f(c_i)\Delta x_i$$

Taking the limit as the norm $\|P\|$ tends to zero, we obtain

$$\int_a^b g(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{N} g(c_i)\Delta x_i \le \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$$ ■

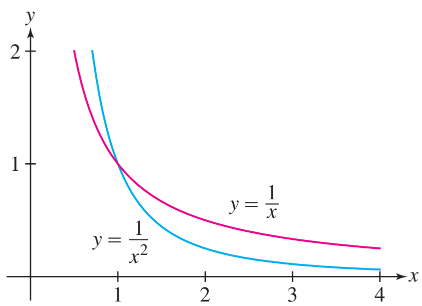


FIGURE 11

■ **EXAMPLE 6** Prove the inequality $\int_1^4 \frac{1}{x^2}\,dx \le \int_1^4 \frac{1}{x}\,dx$.

Solution If $x \ge 1$, then $x^2 \ge x$, and $x^{-2} \le x^{-1}$ (Figure 11). Therefore, the inequality follows from the Comparison Theorem, applied with $g(x) = x^{-2}$ and $f(x) = x^{-1}$.

Suppose there are numbers m and M such that $m \le f(x) \le M$ for x in $[a, b]$. We call m and M **lower** and **upper bounds** for $f(x)$ on $[a, b]$. By the Comparison Theorem,

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \qquad \mathbf{8}$$

This says simply that the integral of f lies between the areas of two rectangles (Figure 12). ■

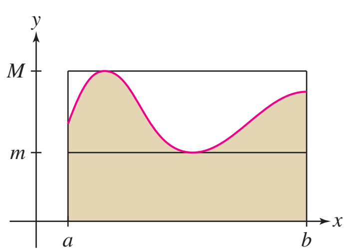


FIGURE 12 The integral $\int_a^b f(x)\,dx$ lies between the areas of the rectangles of heights m and M.

■ **EXAMPLE 7** Prove the inequalities $\frac{3}{4} \le \int_{1/2}^{2} \frac{1}{x}\,dx \le 3$.

Solution Because $f(x) = x^{-1}$ is decreasing (Figure 13), its minimum value on $\left[\frac{1}{2}, 2\right]$ is $m = f(2) = \frac{1}{2}$ and its maximum value is $M = f\left(\frac{1}{2}\right) = 2$. By Eq. (8),

$$\underbrace{\frac{1}{2}\left(2 - \frac{1}{2}\right)}_{m(b-a)} = \frac{3}{4} \le \int_{1/2}^{2} \frac{1}{x}\,dx \le \underbrace{2\left(2 - \frac{1}{2}\right)}_{M(b-a)} = 3$$ ■

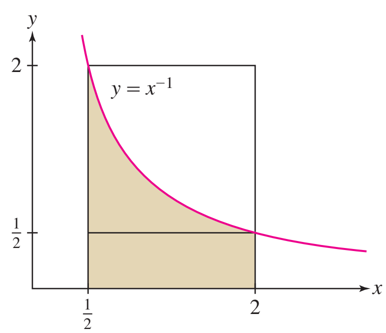


FIGURE 13

5.2 SUMMARY

- A Riemann sum $R(f, P, C)$ for the interval $[a, b]$ is defined by choosing a *partition*

$$P : a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

and *sample points* $C = \{c_i\}$, where $c_i \in [x_{i-1}, x_i]$. Let $\Delta x_i = x_i - x_{i-1}$. Then

$$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i$$

- The maximum of the widths Δx_i is called the norm $\|P\|$ of the partition.
- The *definite integral* is the limit of the Riemann sums (if it exists):

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f, P, C)$$

We say that f is *integrable* over $[a, b]$ if the limit exists.

- Theorem: If f is continuous on $[a, b]$, then f is integrable over $[a, b]$.
- $\int_a^b f(x)\,dx =$ *signed area* of the region between the graph of f and the x-axis.
- Properties of definite integrals:

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx \quad \text{for any constant } C$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \quad \text{for all } a, b, c$$

- Formulas:

$$\int_a^b C\,dx = C(b-a) \quad (C \text{ any constant})$$

$$\int_0^b x\,dx = \frac{1}{2}b^2$$

$$\int_0^b x^2\,dx = \frac{1}{3}b^3$$

- Comparison Theorem: If $f(x) \le g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

If $m \le f(x) \le M$ on $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

5.2 EXERCISES

Preliminary Questions

1. What is $\int_3^5 dx$ [the function is $f(x) = 1$]?

2. Let $I = \int_2^7 f(x)\,dx$, where f is continuous. State whether the following are true or false:

(a) I is the area between the graph and the x-axis over $[2, 7]$.

(b) If $f(x) \ge 0$, then I is the area between the graph and the x-axis over $[2, 7]$.

(c) If $f(x) \le 0$, then $-I$ is the area between the graph of f and the x-axis over $[2, 7]$.

3. Explain graphically: $\int_0^\pi \cos x\,dx = 0$.

4. Which is negative, $\int_{-1}^{-5} 8\,dx$ or $\int_{-5}^{-1} 8\,dx$?

Exercises

In Exercises 1–10, draw a graph of the signed area represented by the integral and compute it using geometry.

1. $\int_{-3}^{3} 2x\,dx$

2. $\int_{-2}^{3} (2x+4)\,dx$

3. $\int_{-2}^{1} (3x+4)\,dx$

4. $\int_{-2}^{1} 4\,dx$

5. $\int_6^8 (7-x)\,dx$

6. $\int_{\pi/2}^{3\pi/2} \sin x\,dx$

7. $\int_0^5 \sqrt{25-x^2}\,dx$

8. $\int_{-2}^{3} |x|\,dx$

9. $\int_{-2}^{2} (2-|x|)\,dx$

10. $\int_{-2}^{5} (3+x-2|x|)\,dx$

11. Calculate $\int_0^{10} (8 - x)\,dx$ in two ways:

(a) As the limit $\lim_{N\to\infty} R_N$

(b) By sketching the relevant signed area and using geometry

12. Calculate $\int_{-1}^{4} (4x - 8)\,dx$ in two ways:

(a) As the limit $\lim_{N\to\infty} R_N$

(b) By using geometry

In Exercises 13 and 14, refer to Figure 14.

13. Evaluate: (a) $\int_0^2 f(x)\,dx$ (b) $\int_0^6 f(x)\,dx$

14. Evaluate: (a) $\int_1^4 f(x)\,dx$ (b) $\int_1^6 |f(x)|\,dx$

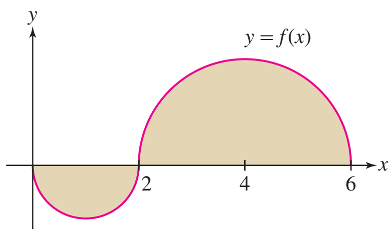


FIGURE 14 The two parts of the graph are semicircles.

In Exercises 15 and 16, refer to Figure 15.

15. Evaluate $\int_0^3 g(t)\,dt$ and $\int_3^5 g(t)\,dt$.

16. Find a, b, and c such that $\int_0^a g(t)\,dt$ and $\int_b^c g(t)\,dt$ are as large as possible.

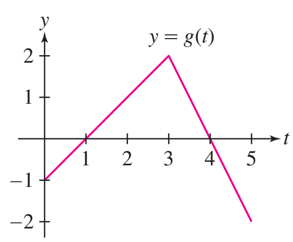


FIGURE 15

17. Describe the partition P and the set of sample points C for the Riemann sum shown in Figure 16. Compute the value of the Riemann sum.

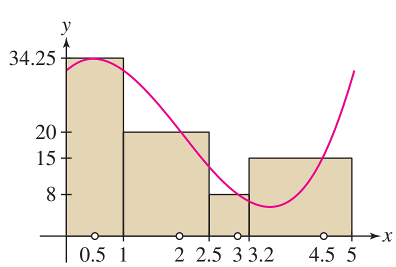


FIGURE 16

18. Compute $R(f, P, C)$ for $f(x) = x^2 + x$ for the partition P and the set of sample points C in Figure 16. [The curve shown is not $f(x) = x^2 + x$.]

In Exercises 19–22, calculate the Riemann sum $R(f, P, C)$ for the given function, partition, and choice of sample points. Also, sketch the graph of f and the rectangles corresponding to $R(f, P, C)$.

19. $f(x) = x$, $P = \{1, 1.2, 1.5, 2\}$, $C = \{1.1, 1.4, 1.9\}$

20. $f(x) = 2x + 3$, $P = \{-4, -1, 1, 4, 8\}$, $C = \{-3, 0, 2, 5\}$

21. $f(x) = x^2 + x$, $P = \{2, 3, 4.5, 5\}$, $C = \{2, 3.5, 5\}$

22. $f(x) = \sin x$, $P = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$, $C = \{0.4, 0.7, 1.2\}$

In Exercises 23–28, sketch the signed area represented by the integral. Indicate the regions of positive and negative area.

23. $\int_0^5 (4x - x^2)\,dx$

24. $\int_{-\pi/4}^{\pi/4} \tan x\,dx$

25. $\int_{\pi}^{2\pi} \sin x\,dx$

26. $\int_0^{3\pi} \sin x\,dx$

27. $\int_{1/2}^{2} \ln x\,dx$

28. $\int_{-1}^{1} \tan^{-1} x\,dx$

In Exercises 29–32, determine the sign of the integral without calculating it. Draw a graph if necessary.

29. $\int_{-2}^{1} x^4\,dx$

30. $\int_{-2}^{1} x^3\,dx$

31. GU $\int_0^{2\pi} x \sin x\,dx$

32. GU $\int_0^{2\pi} \frac{\sin x}{x}\,dx$

In Exercises 33–42, use properties of the integral and the formulas in the summary to calculate the integrals.

33. $\int_0^4 (6t - 3)\,dt$

34. $\int_{-3}^{2} (4x + 7)\,dx$

35. $\int_0^9 x^2\,dx$

36. $\int_2^5 x^2\,dx$

37. $\int_0^1 (u^2 - 2u)\,du$

38. $\int_0^{1/2} (12y^2 + 6y)\,dy$

39. $\int_{-3}^{1} (7t^2 + t + 1)\,dt$

40. $\int_{-3}^{3} (9x - 4x^2)\,dx$

41. $\int_{-a}^{1} (x^2 + x)\,dx$

42. $\int_a^{a^2} x^2\,dx$

In Exercises 43–47, calculate the integral, assuming that

$$\int_0^5 f(x)\,dx = 5, \qquad \int_0^5 g(x)\,dx = 12$$

43. $\int_0^5 (f(x) + g(x))\,dx$

44. $\int_0^5 \left(2f(x) - \frac{1}{3}g(x)\right)\,dx$

45. $\int_5^0 g(x)\,dx$

46. $\int_0^5 (f(x) - x)\,dx$

47. Is it possible to calculate $\int_0^5 g(x)f(x)\,dx$ from the information given?

48. Prove by computing the limit of right-endpoint approximations:

$$\int_0^b x^3\,dx = \frac{b^4}{4} \qquad \boxed{9}$$

In Exercises 49–54, evaluate the integral using the formulas in the summary and Eq. (9).

49. $\int_0^3 x^3\,dx$

50. $\int_1^3 x^3\,dx$

51. $\int_0^3 (x - x^3)\,dx$

52. $\int_0^1 (2x^3 - x + 4)\,dx$

53. $\int_0^1 (12x^3 + 24x^2 - 8x)\,dx$

54. $\int_{-2}^2 (2x^3 - 3x^2)\,dx$

In Exercises 55–58, calculate the integral, assuming that

$$\int_0^1 f(x)\,dx = 1, \qquad \int_0^2 f(x)\,dx = 4, \qquad \int_1^4 f(x)\,dx = 7$$

55. $\int_0^4 f(x)\,dx$

56. $\int_1^2 f(x)\,dx$

57. $\int_4^1 f(x)\,dx$

58. $\int_2^4 f(x)\,dx$

In Exercises 59–62, express each integral as a single integral.

59. $\int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx$

60. $\int_2^9 f(x)\,dx - \int_4^9 f(x)\,dx$

61. $\int_2^9 f(x)\,dx - \int_2^5 f(x)\,dx$

62. $\int_7^3 f(x)\,dx + \int_3^9 f(x)\,dx$

In Exercises 63–66, calculate the integral, assuming that f is integrable and $\int_1^b f(x)\,dx = 1 - b^{-1}$ for all $b > 0$.

63. $\int_1^5 f(x)\,dx$

64. $\int_3^5 f(x)\,dx$

65. $\int_1^6 (3f(x) - 4)\,dx$

66. $\int_{1/2}^1 f(x)\,dx$

67. Explain the difference in graphical interpretation between $\int_a^b f(x)\,dx$ and $\int_a^b |f(x)|\,dx$.

68. Use the graphical interpretation of the definite integral to explain the inequality

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

where f is continuous. Explain also why equality holds if and only if either $f(x) \ge 0$ for all x or $f(x) \le 0$ for all x.

69. Let $f(x) = x$. Find an interval $[a, b]$ such that

$$\left|\int_a^b f(x)\,dx\right| = \frac{1}{2} \quad \text{and} \quad \int_a^b |f(x)|\,dx = \frac{3}{2}$$

70. Evaluate $I = \int_0^{2\pi} \sin^2 x\,dx$ and $J = \int_0^{2\pi} \cos^2 x\,dx$ as follows. First show with a graph that $I = J$. Then prove that $I + J = 2\pi$.

In Exercises 71–74, calculate the integral.

71. $\int_0^6 |3 - x|\,dx$

72. $\int_1^3 |2x - 4|\,dx$

73. $\int_{-1}^1 |x^3|\,dx$

74. $\int_0^2 |x^2 - 1|\,dx$

75. Use the Comparison Theorem to show that

$$\int_0^1 x^5\,dx \le \int_0^1 x^4\,dx, \qquad \int_1^2 x^4\,dx \le \int_1^2 x^5\,dx$$

76. Prove that $\frac{1}{3} \le \int_4^6 \frac{1}{x}\,dx \le \frac{1}{2}$.

77. Prove that $0.0198 \le \int_{0.2}^{0.3} \sin x\,dx \le 0.0296$. *Hint:* Show that $0.198 \le \sin x \le 0.296$ for x in $[0.2, 0.3]$.

78. Prove that $0.277 \le \int_{\pi/8}^{\pi/4} \cos x\,dx \le 0.363$.

79. Prove that $0 \le \int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{\sqrt{2}}{2}$.

80. Find upper and lower bounds for $\int_0^1 \frac{dx}{\sqrt{5x^3 + 4}}$.

81. Suppose that $f(x) \le g(x)$ on $[a, b]$. By the Comparison Theorem, $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. Is it also true that $f'(x) \le g'(x)$ for $x \in [a, b]$? If not, give a counterexample.

82. State whether true or false. If false, sketch the graph of a counterexample.

(a) If $f(x) > 0$, then $\int_a^b f(x)\,dx > 0$.

(b) If $\int_a^b f(x)\,dx > 0$, then $f(x) > 0$.

Further Insights and Challenges

83. Explain graphically: If f is an odd function, then $\int_{-a}^a f(x)\,dx = 0$.

84. Compute $\int_{-1}^1 (\sin x)(\sin^2 x + 1)\,dx$.

85. Let k and b be positive. Show, by comparing the right-endpoint approximations, that

$$\int_0^b x^k\,dx = b^{k+1} \int_0^1 x^k\,dx$$

86. Verify for $0 \le b \le 1$ by interpreting in terms of area:

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2}b\sqrt{1-b^2} + \frac{1}{2}\sin^{-1} b$$

87. Suppose that f and g are continuous functions such that, *for all* a,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} g(x)\,dx$$

Give an *intuitive* argument showing that $f(0) = g(0)$. Explain your idea with a graph.

88. Theorem 4 remains true without the assumption $a \le b \le c$. Verify this for the cases $b < a < c$ and $c < a < b$.

5.3 The Indefinite Integral

In earlier chapters, we have seen how useful it is to be able to find the derivative of a function. But what about the inverse problem? Given the derivative of an unknown function, find the function itself. For example, in physics we may know the velocity $v(t)$ (the derivative) and wish to compute the position $s(t)$ of an object. Since $s'(t) = v(t)$, this amounts to finding a function whose derivative is $v(t)$. A function F whose derivative is f is called an antiderivative of f. Antiderivatives will turn out to be the key to evaluating definite integrals.

DEFINITION Antiderivatives A function F is an antiderivative of f on an open interval (a, b) if $F'(x) = f(x)$ for all x in (a, b).

Examples:

- $F(x) = -\cos x$ is an antiderivative of $f(x) = \sin x$ because for all values of x,

$$F'(x) = \frac{d}{dx}(-\cos x) = \sin x = f(x)$$

- $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ because for all values of x,

$$F'(x) = \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2 = f(x)$$

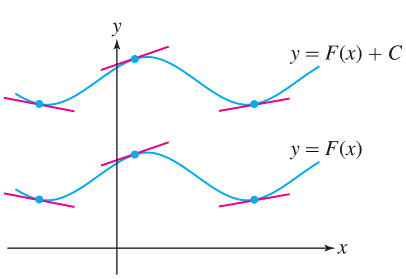
One critical observation is that antiderivatives are not unique. We are free to add a constant C because the derivative of a constant is zero, and so, if $F'(x) = f(x)$, then $(F(x) + C)' = f(x)$. For example, each of the following is an antiderivative of x^2:

$$\frac{1}{3}x^3, \qquad \frac{1}{3}x^3 + 5, \qquad \frac{1}{3}x^3 - 4$$

Are there any antiderivatives of f other than those obtained by adding a constant to a given antiderivative F? Our next theorem says that the answer is no if f is defined on an open interval (a, b).

THEOREM 1 The General Antiderivative Let $y = F(x)$ be an antiderivative of $y = f(x)$ on (a, b). Then every other antiderivative on (a, b) is of the form $y = F(x) + C$ for some constant C.

Proof If $y = G(x)$ is a second antiderivative of $y = f(x)$, set $H(x) = G(x) - F(x)$. Then $H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$. By the Corollary to the Mean Value Theorem in Section 4.3, $H(x)$ must be a constant—say, $H(x) = C$—and therefore $G(x) = F(x) + C$. ■



DF **FIGURE 1** The tangent lines to the graphs of $y = F(x)$ and $y = F(x) + C$ are parallel.

GRAPHICAL INSIGHT The graph of $y = F(x) + C$ is obtained by shifting the graph of $y = F(x)$ vertically by C units. Since vertical shifting moves the tangent lines without changing their slopes, it makes sense that all of the functions $y = F(x) + C$ have the same derivative (Figure 1). Theorem 1 tells us that conversely, if two graphs have parallel tangent lines, then one graph is obtained from the other by a vertical shift.

We often describe the *general* antiderivative of a function in terms of an arbitrary constant C, as in the following example.

EXAMPLE 1 Find two antiderivatives of $f(x) = \cos x$. Then determine the general antiderivative.

Solution The functions $F(x) = \sin x$ and $G(x) = \sin x + 2$ are both antiderivatives of $f(x) = \cos x$. The general antiderivative is $F(x) = \sin x + C$, where C is any constant. ■

The process of finding an antiderivative is called **integration**. We will see why in the next section, when we discuss the connection between antiderivatives and areas under curves given by the Fundamental Theorem of Calculus. Anticipating this result, we begin using the integral sign $\int$, the standard notation for antiderivatives.

The terms "antiderivative" and "indefinite integral" are used interchangeably. In some textbooks, an antiderivative is called a "primitive function."

NOTATION Indefinite Integral The notation

$$\int f(x)\,dx = F(x) + C \quad \text{means that} \quad F'(x) = f(x)$$

We say that $y = F(x) + C$ is the general antiderivative or **indefinite integral** of $y = f(x)$.

The expression $f(x)$ appearing in the integral sign is called the **integrand**. The symbol dx is a *differential*. It is part of the integral notation and serves to indicate the independent variable. The constant C is called the *constant of integration*.

Some indefinite integrals can be evaluated by reversing the familiar derivative formulas. For example, we obtain the indefinite integral of $y = x^n$ by reversing the Power Rule for derivatives.

There are no Product, Quotient, or Chain Rules for integrals. However, we will see that the Product Rule for derivatives leads to an important technique called Integration by Parts (Section 7.1) and the Chain Rule leads to the Substitution Method (Section 5.7).

THEOREM 2 Power Rule for Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{for } n \neq -1$$

Proof We just need to verify that $F(x) = \dfrac{x^{n+1}}{n+1}$ is an antiderivative of $f(x) = x^n$:

$$F'(x) = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{1}{n+1}((n+1)x^n) = x^n$$ ■

In words, the Power Rule for Integrals says that to integrate a power of x, "add one to the exponent and then divide by the new exponent." Here are some examples:

$$\int x^5\,dx = \frac{1}{6}x^6 + C, \qquad \int x^{-9}\,dx = -\frac{1}{8}x^{-8} + C, \qquad \int x^{3/5}\,dx = \frac{5}{8}x^{8/5} + C$$

The Power Rule is not valid for $n = -1$. In fact, for $n = -1$, we obtain the meaningless result

$$\int x^{-1}\,dx = \frac{x^{n+1}}{n+1} + C = \frac{x^0}{0} + C \qquad \text{(meaningless)}$$

Recall, however, that the derivative of the natural logarithm is $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$. This shows that $F(x) = \ln x$ is an antiderivative of $y = \dfrac{1}{x}$. Thus, for $n = -1$, instead of the Power Rule we have

Notice that in integral notation, we treat dx as a movable variable, and thus we write $\int \frac{1}{x}\,dx$ as $\int \frac{dx}{x}$.

$$\int \frac{dx}{x} = \ln x + C$$

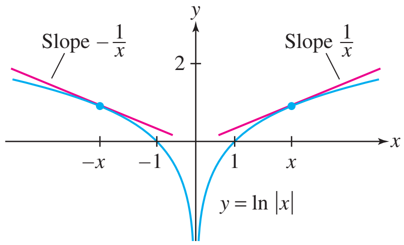


FIGURE 2

This formula is valid for $x > 0$, where $\ln x$ is defined. We would like to have an antiderivative of $y = \frac{1}{x}$ on its full domain, namely on $\{x : x \neq 0\}$. To achieve this end, we extend $F(x)$ to an even function by setting $F(x) = \ln|x|$ (Figure 2). Then $F(x) = F(-x)$, and by the Chain Rule, $F'(x) = -F'(-x)$. For $x < 0$, we obtain

$$\frac{d}{dx}\ln|x| = F'(x) = -F'(-x) = -\frac{1}{-x} = \frac{1}{x}$$

This proves that $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$.

THEOREM 3 Antiderivative of $y = \frac{1}{x}$ The function $F(x) = \ln|x|$ is an antiderivative of $y = \frac{1}{x}$ in the domain $\{x : x \neq 0\}$; that is,

$$\int \frac{dx}{x} = \ln|x| + C \qquad (1)$$

The indefinite integral obeys the usual linearity rules that allow us to integrate "term by term." These rules follow from the linearity rules for the derivative (see Exercise 81).

THEOREM 4 Linearity of the Indefinite Integral

- **Sum Rule:** $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
- **Multiples Rule:** $\int cf(x)\,dx = c\int f(x)\,dx$

■ **EXAMPLE 2** Evaluate $\int (3x^4 - 5x^{2/3} + x^{-3})\,dx$.

Solution We integrate term by term and use the Power Rule:

$$\int (3x^4 - 5x^{2/3} + x^{-3})\,dx = \int 3x^4\,dx - \int 5x^{2/3}\,dx + \int x^{-3}\,dx \qquad \text{(Sum Rule)}$$

$$= 3\int x^4\,dx - 5\int x^{2/3}\,dx + \int x^{-3}\,dx \qquad \text{(Multiples Rule)}$$

$$= 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^{5/3}}{5/3}\right) + \frac{x^{-2}}{-2} + C \qquad \text{(Power Rule)}$$

$$= \frac{3}{5}x^5 - 3x^{5/3} - \frac{1}{2}x^{-2} + C$$

When we break up an indefinite integral into a sum of several integrals as in Example 2, it is not necessary to include a separate constant of integration for each integral.

To check the answer, we verify that the derivative is equal to the integrand:

$$\frac{d}{dx}\left(\frac{3}{5}x^5 - 3x^{5/3} - \frac{1}{2}x^{-2} + C\right) = 3x^4 - 5x^{2/3} + x^{-3}$$ ■

■ **EXAMPLE 3** Evaluate $\int \left(\frac{5}{x} - 3x^{-10}\right) dx$.

Solution Apply Eq. (1) and the Power Rule:

$$\int \left(\frac{5}{x} - 3x^{-10}\right) dx = 5\int \frac{dx}{x} - 3\int x^{-10}\,dx$$

$$= 5\ln|x| - 3\left(\frac{x^{-9}}{-9}\right) + C = 5\ln|x| + \frac{1}{3}x^{-9} + C$$ ■

The differentiation formulas for the trigonometric functions give us the following integration formulas. Each formula can be checked by differentiation.

Basic Trigonometric Integrals

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C, \qquad \int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C, \qquad \int \csc x \cot x\,dx = -\csc x + C$$

Similarly, for any constant $k \neq 0$, the formulas

$$\frac{d}{dx}\sin(kx) = k\cos(kx), \qquad \frac{d}{dx}\cos(kx) = -k\sin(kx)$$

translate to the following indefinite integral formulas:

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$$
$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$$

EXAMPLE 4 Evaluate $\int \big(\sin 8t + 20\cos 9t\big)\,dt$.

Solution

$$\begin{aligned}\int \big(\sin 8t + 20\cos 9t\big)\,dt &= \int \sin 8t\,dt + 20\int \cos 9t\,dt \\ &= -\frac{1}{8}\cos 8t + \frac{20}{9}\sin 9t + C\end{aligned}$$

Integrals Involving e^x

The formula $(e^x)' = e^x$ says that $f(x) = e^x$ is its own derivative. But this means that $f(x) = e^x$ is also *its own antiderivative*. In other words,

$$\int e^x\,dx = e^x + C$$

More generally, for any constant $k \neq 0$,

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

EXAMPLE 5 Evaluate **(a)** $\int (3e^x - 4)\,dx$ and **(b)** $\int 12e^{7-3x}\,dx$.

Solution

(a) $\int (3e^x - 4)\,dx = 3\int e^x\,dx - \int 4\,dx = 3e^x - 4x + C$

(b) $\int 12e^{7-3x}\,dx = 12\int e^7 e^{-3x}\,dx = 12e^7\left(\frac{1}{-3}e^{-3x}\right) = -4e^{7-3x} + C$

Initial Conditions

We can think of an antiderivative as a solution to the **differential equation**

$$\frac{dy}{dx} = f(x) \qquad \boxed{2}$$

In general, a differential equation is an equation relating an unknown function and its derivatives. The unknown in Eq. (2) is a function $y = F(x)$ whose derivative is $f(x)$; that is, $y = F(x)$ is an antiderivative of $y = f(x)$.

Eq. (2) has infinitely many solutions (because the antiderivative is not unique), but we can specify a particular solution by imposing an **initial condition**— that is, by requiring that the solution satisfy $y(x_0) = y_0$ for some fixed values x_0 and y_0. A differential equation with an initial condition is called an **initial value problem**.

An initial condition is like the y-intercept of a line, which determines one particular line among all lines with the same slope. The graphs of the antiderivatives of $y = f(x)$ are all parallel (Figure 1), and the initial condition determines one of them. Sometimes, when the variable is not time, the initial condition is called the boundary condition.

■ **EXAMPLE 6** Solve $\dfrac{dy}{dx} = 4x^7$ subject to the initial condition $y(0) = 4$.

Solution First, find the general antiderivative:

$$y(x) = \int 4x^7\,dx = \frac{1}{2}x^8 + C$$

Then choose C so that the initial condition is satisfied: $y(0) = 0 + C = 4$. This yields $C = 4$, and our solution is $y = \frac{1}{2}x^8 + 4$. ■

■ **EXAMPLE 7** Solve the initial value problem $\dfrac{dy}{dt} = \sin(\pi t)$, $y(2) = 2$.

Solution First, find the general antiderivative:

$$y(t) = \int \sin(\pi t)\,dt = -\frac{1}{\pi}\cos(\pi t) + C$$

Then solve for C by evaluating at $t = 2$:

$$y(2) = -\frac{1}{\pi}\cos(2\pi) + C = 2 \quad \Rightarrow \quad C = 2 + \frac{1}{\pi}$$

The solution of the initial value problem is $y(t) = -\frac{1}{\pi}\cos(\pi t) + 2 + \frac{1}{\pi}$. ■

■ **EXAMPLE 8** A car traveling with velocity 24 m/s begins to slow down at time $t = 0$ s with a constant deceleration of $a = -6$ m/s^2. Find **(a)** the velocity $v(t)$ at time t, and **(b)** the distance traveled before the car comes to a halt.

Solution **(a)** The derivative of velocity is acceleration, so *velocity is the antiderivative of acceleration*:

$$v(t) = \int a\,dt = \int (-6)\,dt = -6t + C$$

The initial condition $v(0) = C = 24$ m/s gives us $v(t) = -6t + 24$.

(b) Position is the antiderivative of velocity, so the car's position in meters is

$$s(t) = \int v(t)\,dt = \int (-6t + 24)\,dt = -3t^2 + 24t + C_1$$

where C_1 is a constant. We are not told where the car is at $t = 0$, so let us set $s(0) = 0$ for convenience, obtaining $C_1 = 0$. With this choice, $s(t) = -3t^2 + 24t$. This is the distance traveled from time $t = 0$.

The car comes to a halt when its velocity is zero, so we solve

$$v(t) = -6t + 24 = 0 \quad \Rightarrow \quad t = 4 \text{ s}$$

The distance traveled before coming to a halt is $s(4) = -3(4^2) + 24(4) = 48$ m. ■

Relation between position, velocity, and acceleration:

$$s'(t) = v(t), \qquad s(t) = \int v(t)\,dt$$

$$v'(t) = a(t), \qquad v(t) = \int a(t)\,dt$$

5.3 SUMMARY

- F is called an *antiderivative* of f if $F'(x) = f(x)$.
- Any two antiderivatives of f on an interval (a, b) differ by a constant.
- The general antiderivative is denoted by the indefinite integral:

$$\int f(x)\,dx = F(x) + C$$

- Integration formulas:

$$\int 0\,dx = C$$

$$\int k\,dx = kx + C \qquad (k \neq 0)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C \qquad (k \neq 0)$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C \qquad (k \neq 0)$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C \qquad (k \neq 0)$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

- To solve an initial value problem $\dfrac{dy}{dx} = f(x)$, $y(x_0) = y_0$, first find the general antiderivative $y = F(x) + C$. Then determine C using the initial condition $F(x_0) + C = y_0$.

5.3 EXERCISES

Preliminary Questions

1. Find an antiderivative of the function $f(x) = 0$.

2. Is there a difference between finding the general antiderivative of a function f and evaluating $\int f(x)\,dx$?

3. Jacques was told that f and g have the same derivative, and he wonders whether $f(x) = g(x)$. Does Jacques have sufficient information to answer his question?

4. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Which of the following statements are true? Explain.

(a) If $f = g$, then $F = G$.

(b) If F and G differ by a constant, then $f = g$.

(c) If f and g differ by a constant, then $F = G$.

5. Is $y = x$ a solution of the following initial value problem?

$$\frac{dy}{dx} = 1, \qquad y(0) = 1$$

Exercises

In Exercises 1–8, find the general antiderivative of f and check your answer by differentiating.

1. $f(x) = 18x^2$

2. $f(x) = x^{-3/5}$

3. $f(x) = 2x^4 - 24x^2 + 12x^{-1}$

4. $f(x) = 9x + 15x^{-2}$

5. $f(x) = 2\cos x - 9\sin x$

6. $f(x) = 4x^7 - 3\cos x$

7. $f(x) = 12e^x - 5x^{-2}$

8. $f(x) = e^x - 4\sin x$

9. Match functions (a)–(d) with their antiderivatives (i)–(iv).

(a) $f(x) = \sin x$

(b) $f(x) = x\sin(x^2)$

(c) $f(x) = \sin(1 - x)$

(d) $f(x) = x\sin x$

(i) $F(x) = \cos(1 - x)$

(ii) $F(x) = -\cos x$

(iii) $F(x) = -\frac{1}{2}\cos(x^2)$

(iv) $F(x) = \sin x - x\cos x$

In Exercises 10–39, evaluate the indefinite integral.

10. $\int (9x + 2)\,dx$

11. $\int (4 - 18x)\,dx$

12. $\int x^{-3}\,dx$

13. $\int t^{-6/11}\,dt$

14. $\int (5t^3 - t^{-3})\,dt$

15. $\int (18t^5 - 10t^4 - 28t)\,dt$

16. $\int 14s^{9/5}\,ds$

17. $\int (z^{-4/5} - z^{2/3} + z^{5/4})\,dz$

18. $\int \frac{3}{2}\,dx$

19. $\int \frac{1}{\sqrt[3]{x}}\,dx$

20. $\int \frac{dx}{x^{4/3}}$

21. $\int \frac{36\,dt}{t^3}$

22. $\int x(x^2 - 4)\,dx$

23. $\int (t^{1/2} + 1)(t + 1)\,dt$

24. $\int \frac{12 - z}{\sqrt{z}}\,dz$

25. $\int \frac{x^3 + 3x - 4}{x^2}\,dx$

26. $\int \left(\frac{1}{3}\sin x - \frac{1}{4}\cos x\right)dx$

27. $\int 12\sec x\tan x\,dx$

28. $\int (\theta + \sec^2\theta)\,d\theta$

29. $\int (\csc t\cot t)\,dt$

30. $\int \sin(7x)\,dx$

31. $\int \sec^2(-3\theta)\,d\theta$

32. $\int (\theta - \cos(-\theta))\,d\theta$

33. $\int 25\sec^2(3z)\,dz$

34. $\int \sec x\tan x\,dx$

35. $\int \left(\cos(3\theta) - \frac{1}{2}\sec^2\left(\frac{\theta}{4}\right)\right)d\theta$

36. $\int \left(\frac{4}{x} - e^x\right)dx$

37. $\int (3e^{5x})\,dx$

38. $\int e^{3t-4}\,dt$

39. $\int (8x - 4e^{5-2x})\,dx$

40. In Figure 3, is graph (A) or graph (B) the graph of an antiderivative of $y = f(x)$?

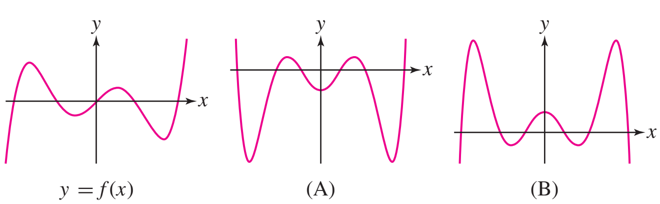


FIGURE 3

41. In Figure 4, which of graphs (A), (B), and (C) is *not* the graph of an antiderivative of $y = f(x)$? Explain.

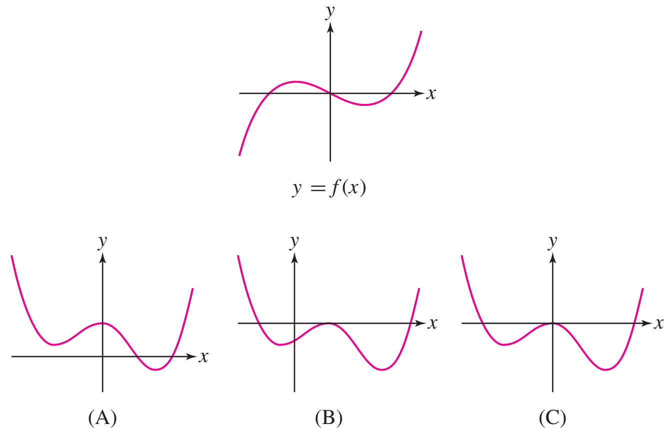


FIGURE 4

42. Show that $F(x) = \frac{1}{3}(x + 13)^3$ is an antiderivative of $f(x) = (x + 13)^2$.

In Exercises 43–46, verify by differentiation.

43. $\int (x + 13)^6\,dx = \frac{1}{7}(x + 13)^7 + C$

44. $\int (x + 13)^{-5}\,dx = -\frac{1}{4}(x + 13)^{-4} + C$

45. $\int (4x + 13)^2\,dx = \frac{1}{12}(4x + 13)^3 + C$

46. $\int (ax + b)^n\,dx = \frac{1}{a(n + 1)}(ax + b)^{n+1} + C \quad (\text{for } n \neq -1)$

In Exercises 47–62, solve the initial value problem.

47. $\frac{dy}{dx} = x^3, \quad y(0) = 4$

48. $\frac{dy}{dt} = 3 - 2t, \quad y(0) = -5$

49. $\frac{dy}{dt} = 2t + 9t^2, \quad y(1) = 2$

50. $\frac{dy}{dx} = 8x^3 + 3x^2, \quad y(2) = 0$

51. $\frac{dy}{dt} = \sqrt{t}, \quad y(1) = 1$

52. $\frac{dz}{dt} = t^{-3/2}, \quad z(4) = -1$

53. $\frac{dy}{dx} = (3x + 2)^3, \quad y(0) = 1$

54. $\frac{dy}{dt} = (4t + 3)^{-2}, \quad y(1) = 0$

55. $\frac{dy}{dx} = \sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

56. $\frac{dy}{dz} = \sin 2z, \quad y\left(\frac{\pi}{4}\right) = 4$

57. $\frac{dy}{dx} = \cos 5x, \quad y(\pi) = 3$

58. $\frac{dy}{dx} = \sec^2 3x, \quad y\left(\frac{\pi}{4}\right) = 2$

59. $\frac{dy}{dx} = e^x, \quad y(2) = 0$

60. $\frac{dy}{dt} = e^{-t}, \quad y(0) = 0$

61. $\frac{dy}{dt} = 9e^{12-3t}, \quad y(4) = 7$

62. $\frac{dy}{dt} = t + 2e^{t-9}, \quad y(9) = 4$

In Exercises 63–69, first find f' and then find f.

63. $f''(x) = 12x, \quad f'(0) = 1, \quad f(0) = 2$

64. $f''(x) = x^3 - 2x, \quad f'(1) = 0, \quad f(1) = 2$

65. $f''(x) = x^3 - 2x + 1, \quad f'(0) = 1, \quad f(0) = 0$

66. $f''(x) = x^3 - 2x + 1, \quad f'(1) = 0, \quad f(1) = 4$

67. $f''(t) = t^{-3/2}, \quad f'(4) = 1, \quad f(4) = 4$

68. $f''(\theta) = \cos\theta, \quad f'\left(\frac{\pi}{2}\right) = 1, \quad f\left(\frac{\pi}{2}\right) = 6$

69. $f''(t) = t - \cos t, \quad f'(0) = 2, \quad f(0) = -2$

70. Show that $F(x) = \tan^2 x$ and $G(x) = \sec^2 x$ have the same derivative. What can you conclude about the relation between F and G? Verify this conclusion directly.

71. A particle located at the origin at $t = 1$ s moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. State the differential equation with its initial condition satisfied by the position $s(t)$ of the particle, and find $s(t)$.

72. A particle moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. Find the particle's position $s(t)$, assuming that $s(2) = 4$ m.

73. A water balloon is dropped from a high building. It falls for 5 s before hitting the ground. Determine the velocity it is traveling when it is about to hit the ground, assuming an acceleration due to gravity of -9.8 m/s^2 and no wind resistance.

74. A hammer is dropped and it falls for 2 s before hitting the ground. Determine how far it falls, assuming an acceleration due to gravity of -9.8 m/s^2 and no wind resistance.

75. A mass oscillates at the end of a spring. Let $s(t)$ be the displacement of the mass from the equilibrium position at time t. Assuming that the mass is located at the origin at $t = 0$ and has velocity $v(t) = \sin(\pi t/2)$ m/s, state the differential equation with initial condition satisfied by $s(t)$, and find $s(t)$.

76. Beginning at $t = 0$ s with initial velocity 4 m/s, a particle moves in a straight line with acceleration $a(t) = 3t^{1/2}$ m/s^2. Find the distance traveled after 25 s.

77. A car traveling 25 m/s begins to decelerate at a constant rate of 4 m/s^2. After how many seconds does the car come to a stop and how far will the car have traveled during its deceleration before stopping?

78. At time $t = 1$ s, a particle is traveling at 72 m/s and begins to decelerate at the rate $a(t) = -t^{-1/2}$ until it stops. How far does the particle travel during its deceleration before stopping?

79. A 900-kg rocket is released from a space station. As it burns fuel, the rocket's mass decreases and its velocity increases. Let $v(m)$ be the velocity (in meters per second) as a function of mass m. Find the velocity when $m = 729$ kg if $dv/dm = -50m^{-1/2}$. Assume that $v(900) = 0$ m/s.

80. As water flows through a tube of radius $R = 10$ cm, the velocity v of an individual water particle depends only on its distance r from the center of the tube. The particles at the walls of the tube have zero velocity and $dv/dr = -0.06r$. Determine $v(r)$.

81. Verify the linearity properties of the indefinite integral stated in Theorem 4.

Further Insights and Challenges

82. Find constants c_1 and c_2 such that $F(x) = c_1 x \sin x + c_2 \cos x$ is an antiderivative of $f(x) = x \cos x$.

83. Find constants c_1 and c_2 such that $F(x) = c_1 x e^x + c_2 e^x$ is an antiderivative of $f(x) = xe^x$.

84. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Is it true that $y = F(x)G(x)$ is an antiderivative of $y = f(x)g(x)$? Confirm or provide a counterexample.

85. Suppose that $F'(x) = f(x)$.

(a) Show that $y = \frac{1}{2}F(2x)$ is an antiderivative of $y = f(2x)$.

(b) Find the general antiderivative of $y = f(kx)$ for $k \neq 0$.

86. Find an antiderivative for $f(x) = |x|$.

87. Using Theorem 1, prove that if $F'(x) = f(x)$, where f is a polynomial of degree $n - 1$, then F is a polynomial of degree n. Then prove that if g is any function such that $g^{(n)}(x) = 0$, then g is a polynomial of degree at most n.

88. Show that $F(x) = \dfrac{x^{n+1} - 1}{n + 1}$ is an antiderivative of $y = x^n$ for $n \neq -1$. Then use L'Hôpital's Rule to prove that

$$\lim_{n \to -1} F(x) = \ln x$$

In this limit, x is fixed and n is the variable. This result shows that, although the Power Rule breaks down for $n = -1$, the antiderivative of $y = x^{-1}$ is a limit of antiderivatives of $y = x^n$ as $n \to -1$.

5.4 The Fundamental Theorem of Calculus, Part I

The FTC was first stated clearly by Isaac Newton in 1666, although other mathematicians, including Newton's teacher Isaac Barrow, had discovered versions of it earlier.

Having so far introduced both derivatives and integrals, a very reasonable question is why they appear together in this topic called Calculus. The answer is the Fundamental Theorem of Calculus (FTC), which is one of the most important theorems in all of mathematics. This foundational result reveals an unexpected connection between the two main operations of calculus: differentiation and integration. The theorem has two parts. Although they are closely related, we discuss them in separate sections to emphasize the different ways they are used. The first part of the Fundamental Theorem of Calculus will allow us to compute definite integrals without having to take limits of Riemann sums.

← REMINDER

*F is called an **antiderivative** of f if $F'(x) = f(x)$. We say also that F is an **indefinite integral** of f, and we use the notation*

$$\int f(x)\,dx = F(x) + C$$

To explain FTC I, recall a result from Example 5 of Section 5.2:

$$\int_4^7 x^2\,dx = \left(\frac{1}{3}\right)7^3 - \left(\frac{1}{3}\right)4^3 = 93$$