Math 111 WS 10, Properties of Logarithms

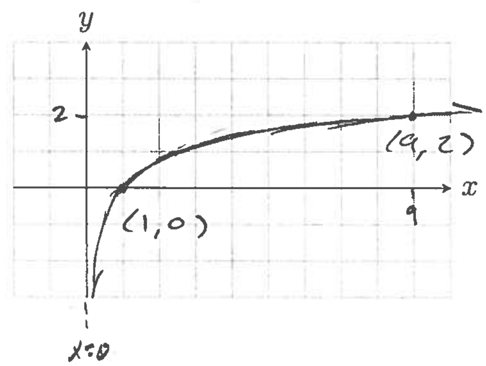
Name: Solutions

1. Suppose $f(x) = \log_a(x)$ where $f(1) = 0$ and $f(9) = 2$. Sketch a graph of the situation and then determine the value of a.

$\log_a(9) = 2 \quad \Rightarrow \quad a^2 = 9$

$\Rightarrow \quad a = 3$

$f(x) = \log_3(x)$



2. Evaluate the following expressions.

a. $\log_a(1) = 0$

b. $\log_a(a) = 1$

c. $2^{\log_2(\pi)} = \pi$

d. $\log_{0.2}\left(0.2^{-\sqrt{2}}\right) = -\sqrt{2}$

e. $\ln e^{kt} = kt$

f. $e^{\ln(1/2)\cdot t/3} = \left(e^{\ln(1/2)}\right)^{t/3}$

$= (1/2)^{t/3} = 2^{-t/3}$

3. Prove that $\log_a(MN) = \log_a(M) + \log_a(N)$

$\log_a(MN) = \log_a\left(a^{\log_a(M)} \cdot a^{\log_a(N)}\right)$

$= \log_a\left(a^{\log_a(M) + \log_a(N)}\right)$

$= \log_a(M) + \log_a(N)$

$\log_a(M/N) = \log_a\left(\dfrac{a^{\log_a(M)}}{a^{\log_a(N)}}\right)$

$= \log_a\left(a^{\log_a(M) - \log_a(N)}\right)$

$= \log_a(M) - \log_a(N)$

4. Prove that $\log_a(M^r) = r\log_a(M)$

$$\log_a(M^r) = \log_a(\overbrace{M \cdot M \cdot \ldots \cdot M}^{r \text{ times}})$$
$$= \log_a(M) + \ldots + \log_a(M)$$
$$= r\log_a(M)$$

5. Write the following expressions as a sum or difference of logarithms, pulling any exponents out of the logarithm by the end of the process.

a. $\log_a\left(x\sqrt{x^2+1}\right)$, $x > 0$

$$\log_a(x\sqrt{x^2+1})$$
$$= \log_a(x) + \log_a((x^2+1)^{1/2})$$
$$= \log_a(x) + \frac{1}{2}\log_a(x^2+1)$$

c. $\log_7\left(x^5 \cdot (x+3)^{2/3}\right)$, $x > 0$

$$\log_7(x^5 \cdot (x+3)^{2/3})$$
$$= \log_7(x^5) + \log_7((x+3)^{2/3})$$
$$= 5\log_7(x) + \frac{2}{3}\log_7(x+3)$$

b. $\ln\left(\dfrac{x^2}{(x-1)^3}\right)$, $x > 1$

$$\ln\left(\frac{x^2}{(x-1)^3}\right)$$
$$= \ln(x^2) - \ln((x-1)^3)$$
$$= 2\ln(x) - 3\ln(x-1)$$

d. $\log_a\left(\dfrac{\sqrt{x^2+1}}{x^3(x+1)^4}\right)$, $x > 0$

$$\log_a\left(\frac{\sqrt{x^2+1}}{x^3(x+1)^4}\right)$$
$$= \log_a((x^2+1)^{1/2}) - \log_a(x^3(x+1)^4)$$
$$= \frac{1}{2}\log_a(x^2+1) - \left[\log_a(x^3) + \log_a(x+1)^4\right]$$
$$= \frac{1}{2}\log_a(x^2+1) - 3\log_a(x) - 4\log_a(x+1)$$

6. Write each of the following sums or differences of logarithms as a single logarithm.

a. $\log_a(7) + 4\log_a(3)$

$= \log_a(7) + \log_a(3^4)$

$= \log_a(7 \cdot 3^4)$

c. $\log_a(x) + \log_a(9) + \log_a(x^2+1) - \log_a(5)$

$= \log_a(9x) + \log_a\left(\frac{x^2+1}{5}\right)$

$= \log_a\left(\frac{9x^3+9x}{5}\right)$

b. $\frac{2}{3}\ln(8) - \ln(5^2 - 1)$

$= \ln(8^{2/3}) - \ln(24)$

$= \ln(4) - \ln(24)$

$= \ln\left(\frac{1}{6}\right)$

d. $21\log_3(\sqrt[3]{x}) + \log_3(9x^2) - \log_3(9)$

$= \log_3(x^7) + \log_3(x^2)$

$= \log_3(x^9)$

7. Approximate $\log_2(7)$ by first rewriting it as an exponential equation and then taking the ln of both sides.

$\log_2(7) = x \quad \Rightarrow \quad 2^x = 7$

$\ln(2^x) = \ln(7)$

$x\ln(2) = \ln(7)$

$x = \frac{\ln(7)}{\ln(2)}$

OR $\quad 2^x = 7$

$\log_3(2^x) = \log_3(7)$

$x\log_3(2) = \log_3(7)$

$x = \log_3(7)/\log_3(2)$

$\Rightarrow \quad \log_2(7) = \frac{\ln(7)}{\ln(2)} = \frac{\log_3(7)}{\log_3(2)} = \frac{\log_8(7)}{\log_8(2)} = \frac{\log_a(7)}{\log_a(2)}$

8. Prove the change of base formula $\log_a(M) = \dfrac{\log_b(M)}{\log_b(a)}$ where b is any number you choose so long as $a \neq 1$, $b \neq 1$ and a, b, and M are all positive real numbers.

Set $\log_a(M) = y$

so $a^y = M$

Then $\log_b(a^y) = \log_b(M)$

so $y \log_b(a) = \log_b(M)$

& $y = \dfrac{\log_b(M)}{\log_b(a)}$ i.e. $\log_a(M) = \dfrac{\log_b(M)}{\log_b(a)}$

9. Approximate the following values by first applying the change of base formula.

a. $\log_5(89) = \dfrac{\ln(89)}{\ln(5)} \approx 2.789$

OR $= \dfrac{\log(89)}{\log(5)} \approx 2.789$

b. $\log_{\sqrt{2}}(\sqrt{5}) = \dfrac{\ln(\sqrt{5})}{\ln(\sqrt{2})} \approx 2.322$

OR $= \dfrac{\log(\sqrt{5})}{\log(\sqrt{2})} \approx 2.322$