Math 251 In Class Worksheet 13

a. The position of a particle is given by the equation

$v(t) = \text{velocity} = s'(t)$

$a(t) = \text{acceleration} = v'(t)$

$$s = f(t) = t^3 - 6t^2 + 9t$$

↑position

where t is measured in seconds and s in meters.

a. Find the velocity at time t.

$v(t) = 3t^2 - 12t + 9$

b. What is the velocity after 2 seconds?

$v(2) = 3(2)^2 - 12(2) + 9 = -3\,m/s$

c. When is the particle at rest?

$0 = 3(t)^2 - 12(t) + 9$

$0 = 3(t^2 - 4t + 3)$

$0 = 3(t-3)(t-1)$

$t = 3s \quad t = 1s$

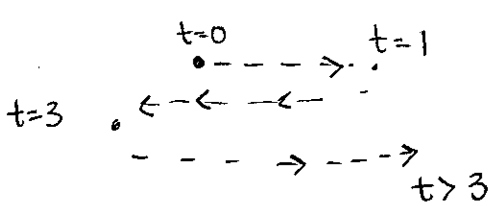
d. When is the particle moving forward?

When is the particle moving to the right?

When $v(t) > 0$, so when

$t < 1$ and $t > 3$

e. Draw a diagram to represent the motion of the particle.



f. Find the total distance traveled by the particle during the first five seconds.

$s(0) = 0m \quad s(1) = 4m$

$s(3) = 0m \quad s(5) = 20m$

$0 - 1s = 4m$

$1 - 3s = 4m$

$3 - 5s = 20m$

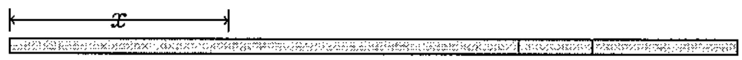
Total distance traveled $= 28m$

g. Find the acceleration at time t and after 4 seconds.

$a = v'(t) = 6t - 12$

$a(4) = 6(4) - 12 = 12\,m/s^2$

b. If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/l$). Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$, as shown in the below figure.



This part of the rod has mass $f(x)$. x_1 x_2

Suppose that $m = f(x) = \sqrt{x}$ is measured in kilograms and that x is measured in meters.

a. Determine the average density of the part of the rod given by $1 \leq x \leq 1.2$.

$m(1) = 1\text{ kg}$ $\quad m(1.2) = \sqrt{1.2} = 1.094\text{ kg}$

avg density $(\rho) = \dfrac{m_2 - m_1}{x_2 - x_1} = \dfrac{\sqrt{1.2} - \sqrt{1}}{1.2\text{ m} - 1\text{ m}} = 0.477\text{ kg/m}$

b. Determine the density right at $x = 1$.

instantaneous density is $m'(x) = \sqrt{x}$ $\quad m'(x) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}$

$m'(x) = \frac{1}{2}(1)^{-1/2} = \frac{1}{2}\text{ kg/m}$

c. One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure - namely, the derivative $\frac{dV}{dP}$. As P increases, V decreases, so $\frac{dV}{dP} < 0$. The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}$$

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

a. Suppose the volume V (in cubic meters) of a sample of air at 25°C was found to be related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}.$$

Find the rate of change of V with respect to P when $P = 50$ kPa.

$V = 5.3P^{-1}$ $\qquad \frac{dV}{dP} = \frac{d}{dP}\left(5.3P^{-1}\right) = \frac{-5.3}{P^2}$

$$\left.\frac{dV}{dP}\right|_{P=50\,kPa} = \frac{-5.3}{50^2} = -0.0212\, m^3/kPa$$

rate of Volume change at 50kPa

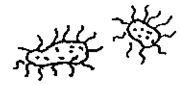
b. Determine the compressibility at that pressure.

$\beta = -\frac{1}{V}\frac{dV}{dP} = \frac{-dV}{dP}\Big/V$ $\quad$ *rate of change per cubic meter

$\frac{dV}{dP} = -\frac{5.3}{P^2}$ $\qquad \left.\beta\right|_{P=50kPa} = \left.-\frac{dV}{dP}\right|_{P=50kPa} \Big/ \left.V\right|_{P=50kPa}$

$V = \frac{5.3}{P}$ $\qquad = +0.0212\, m^3/kPa \cdot 0.11 m^3 = \frac{1}{50}\, m^3/kPa / m^3$

$\left.V\right|_{P=50kPa} = \frac{5.3}{50} = 0.11 m^3$

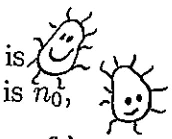


d. Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **instantaneous rate of growth** is obtained from this average rate of growth by letting the time period Δt approach 0:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$



a. Suppose that by sampling the population of a bacteria at certain intervals it is determined that the population doubles every hour. If the initial population is n_0, determine the population as a function of time.

initial population $= n_0$ $\quad f(0t) = n_0 \quad f(1) = 2n_0 \quad f(2) = 2 \cdot 2n_0 \; (n_0 \cdot 2^2)$

exponential $\quad f(t) = n_0 \cdot 2^t$

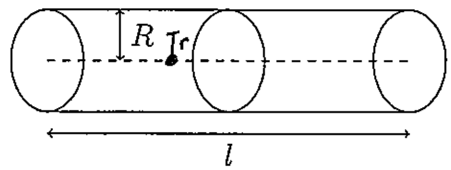
b. If we start with an initial population of $n_0 = 100$ bacteria, determine the growth rate after 4 hours.

$f(t) = n_0 \cdot 2^t \qquad n_0 = 100$

$$\frac{dn}{dt} = \frac{d}{dt}\left(n_0 \cdot 2^t\right) = \frac{d}{dt}\left(100 \cdot 2^t\right) = 100 \cdot \ln(2) \cdot 2^t$$

$$\left.\frac{dn}{dt}\right|_{t=4hr} = 100(\ln 2)\, 2^4 = 1109.035 \text{ bacteria/hr}$$

e. When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l.



Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relation ship between v and r is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This law states that

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$. η =constant

a. Determine $\dfrac{dv}{dr}$

$$\frac{dv}{dr} = \frac{P}{4\eta l}(-2r)$$

b. For one of the smaller human arteries we can take $\eta = 0.027$, $R = 0.008$cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2. Determine $v(r)$ for these conditions.

$$v(r) = \frac{4000}{4(0.027)(2)}(0.008\text{cm}^2 - r^2)$$

$$v(r) = 1.\overline{185} - 18518.5\overline{185}r^2$$

c. What is the speed of the blood at $r = 0.002$ cm?

$$v(0.002) = 1.\overline{1}$$

d. What is the instantaneous change of velocity with respect to r when $r = 0.002$?

$$\frac{dv}{dr} = \frac{4000}{4(0.027)(2)}(-2(0.002)) = -74.0741 \text{cm/s}$$