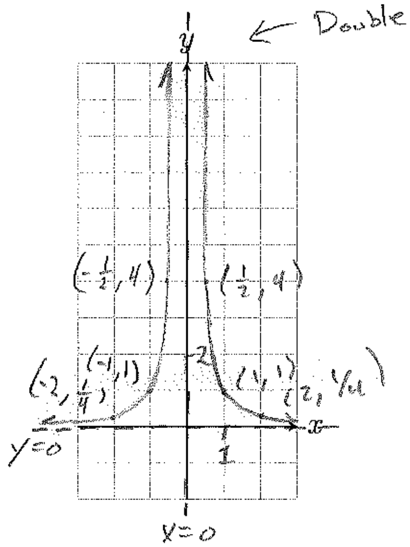
Math 111 WS 14, Rational Functions Name: Solutions

Definition: A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions.

1. Graph and analyze the function $H(x) = \dfrac{1}{x^2}$

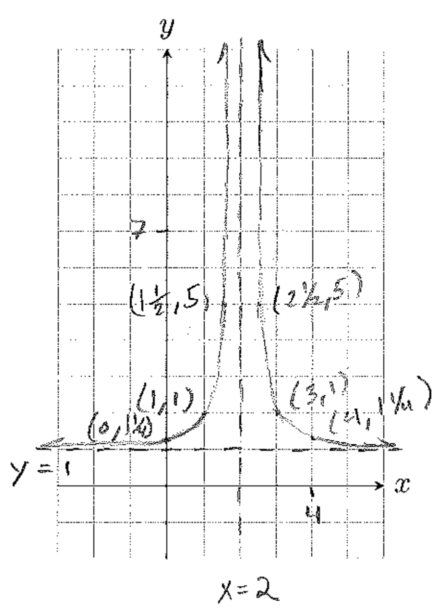


$H(-x) = \dfrac{1}{(-x)^2} = \dfrac{1}{x^2}$ H is even

$D = (-\infty, 0) \cup (0, \infty)$

$R = (0, \infty)$

2. Use transformations to graph $R(x) = \dfrac{1}{(x-2)^2} + 1$.



Keys:	right 2	up 1
$(-\frac{1}{2}, 4)$	$(1\frac{1}{2}, 4)$	$(1\frac{1}{2}, 5)$
$(-1, 1)$	$(1, 1)$	$(1, 2)$
$(-2, \frac{1}{4})$	$(0, \frac{1}{4})$	$(0, 1\frac{1}{4})$
$x = 0$	$x = 2$	$x = 2$
$(2, \frac{1}{4})$	$(4, \frac{1}{4})$	$(4, 1\frac{1}{4})$
$(1, 1)$	$(3, 1)$	$(3, 2)$
$(\frac{1}{2}, 4)$	$(2\frac{1}{2}, 4)$	$(2\frac{1}{2}, 5)$
$y = 0$	$y = 0$	$y = 1$

3. Find the vertical asymptotes, if any, of the graph of each rational function given.

a. $F(x) = \dfrac{x+3}{x-1}$

V.A.: $x=1$

c. $H(x) = \dfrac{x^2}{x^2+1}$

$x^2+1=0$ has no solutions

No V.A.s

b. $R(x) = \dfrac{x}{x^2-4} = \dfrac{x}{(x-2)(x+2)}$

V.A.: $x=2$ + $x=-2$

d. $G(x) = \dfrac{x^2-9}{x^2+4x-21} = \dfrac{(x-3)(x+3)}{\underbrace{(x-3)}_{\text{hole}}\underbrace{(x+7)}_{\text{V.A.}}}$

V.A. @ $x=-7$

4. Determine if the following functions have a horizontal asymptote, an oblique asymptote, or neither. If it does have one or the other, find it.

a. $R(x) = \dfrac{x-12}{4x^2+x+1}$ ← deg 1 ← deg 2

H.A. @ $y=0$

since denominator grows faster than the numerator

b. $H(x) = \dfrac{3x^4-x^2}{x^3-x^2+1}$

$$\begin{array}{r|l} & 3x + 3 \\ x^3-x^2+0x+1 & 3x^4+0x^3-x^2+0x+0 \\ & -(3x^4-3x^3+0x^2+3x+0) \\ & 3x^3-x^2+3x+0 \\ & -(3x^3-3x^2+0x+3) \\ & \underbrace{2x^2+3x-3}_{\text{remainder}} \end{array}$$

$= \underbrace{3x+3}_{\text{Oblique Asymptote}} + \underbrace{\dfrac{2x^2+3x-3}{x^3-x^2+1}}_{\text{goes to zero as } x \to \pm\infty}$

Oblique Asymptote @ $y=3x+3$

c. $K(x) = \dfrac{8x^2 - x + 2}{4x^2 - 1}$ ← same degrees so growing similarly

H.A. @ $y = \frac{8}{4} = 2$

5. For the function $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$ determine the domain, any vertical asymptotes and any horizontal or oblique asymptotes. Sketch a graph of the function.

$R(x) = \dfrac{(2x-1)(x-2)}{\underbrace{(x+2)}_{\text{asymptote}}\underbrace{(x-2)}_{\text{hole}}}$

1st deg → asymptote hole @ $x = 2$

$2x^2 - 5x + 2$

$= 2x^2 - x - 4x + 2$

$= x(2x-1) - 2(2x-1) = (2x-1)(x-2)$

4; $-1 \cdot -4$; $-1 - 4 = -5$

$R(x) = \dfrac{2x-1}{x+2}$

H.A @ $y = 2$

V.A. @ $x = -2$

$D = \{x \mid x \neq \pm 2\}$

$R(0) = -\frac{1}{2}$

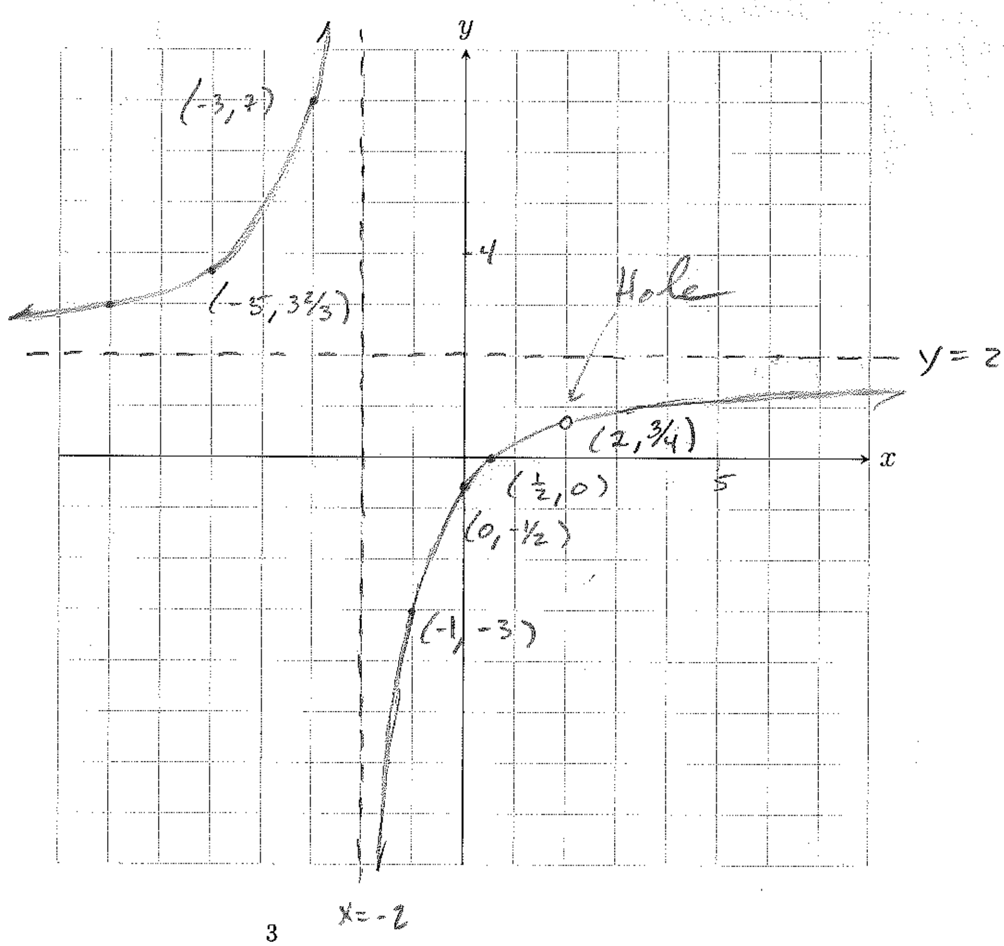
$R(-1) = -3$

$R(-3) = 7$

$R(-5) = \frac{11}{3} = 3\frac{2}{3}$

$R(-7) = 3$

x-int: $0 = 2x - 1 \Rightarrow x = \frac{1}{2}$



$R = \{y \mid y \neq 0\}$

6. Analyze and then graph the rational function $f(x) = \dfrac{x+2}{x^2 - 5x - 6}$.

$f(x) = \dfrac{x+2}{(x-6)(x+1)}$

H.A.: $y = 0$

x-ints:

$f(0) = -1/3$

$0 = x + 2 \Rightarrow x = -2$

V.A.: $x = 6$ & $x = -1$

both 1st degree

x-int: $(-2, 0)$

y-int: $(0, -1/3)$

$f(-6) = \dfrac{-4}{60} = -\dfrac{1}{15}$

$f(-4) = \dfrac{-2}{30} = -1/15$

$D = \{x \mid x \neq -1, 6\}$

$f(-5) = \dfrac{-3}{44}$

Range needs a calculator

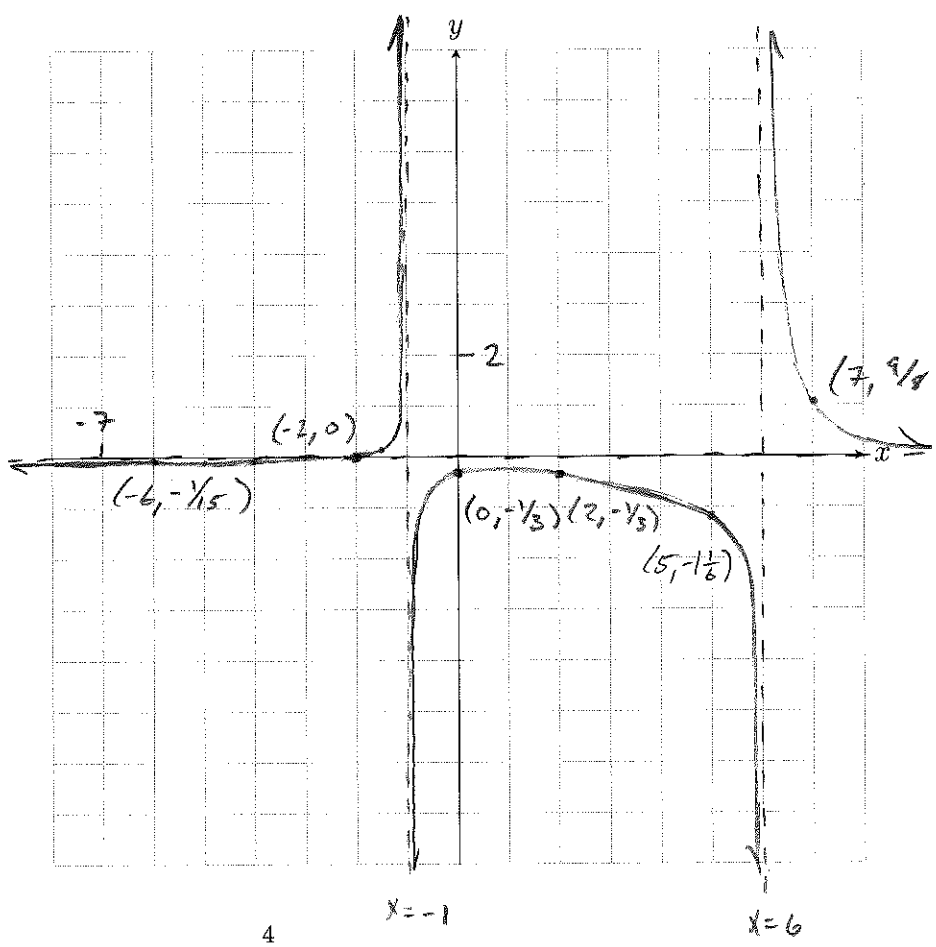
$f(-3) = \dfrac{-1}{18}$

$f(-3/2) = \dfrac{1/2}{\frac{-15}{2} \cdot \frac{-1}{2}} = \dfrac{2}{15}$

$f(2) = \dfrac{4}{-12} = -\dfrac{1}{3}$

$f(5) = -\dfrac{7}{6}$

$f(7) = \dfrac{9}{8}$



7. Analyze and then graph the rational function $R(x) = \dfrac{x^2 - 1}{x}$. ← even ← odd } odd

$$R(x) = \frac{x^2-1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$$ ← V.A. @ $x=0$ 1st deg.

↑ $y=x$ oblique asymptote

No y-int.

$D = \{x \mid x \neq 0\}$

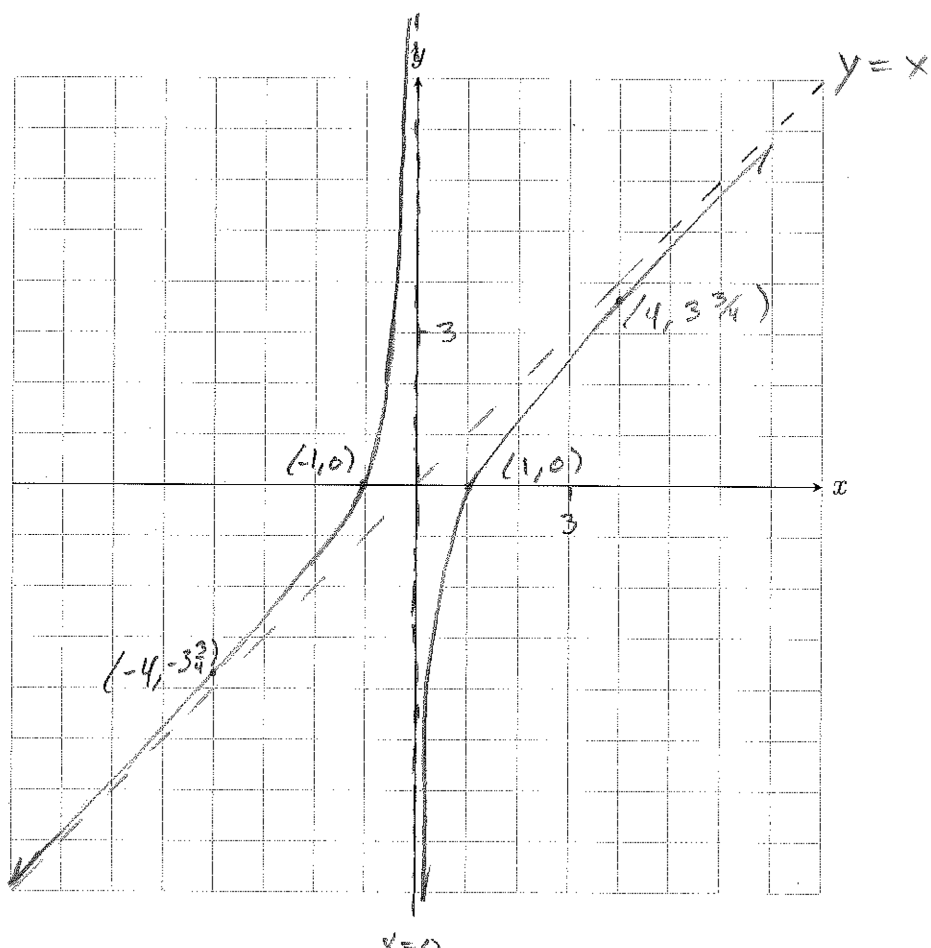
x-ints: $0 = x^2 - 1 = (x-1)(x+1)$

$x = \pm 1$

$(1,0)$ + $(-1,0)$

$R(4) = \frac{15}{4} = 3\frac{3}{4}$

$R(-4) = -3\frac{3}{4}$



8. Analyze and then graph the rational function $R(x) = \dfrac{x^4+1}{x^2}$.

even even doesn't factor

even

No H.A.

No oblique either!

V.A. @ $x=0$ 2nd deg!

no y-int.

x-ints: $0 = x^4 + 1$ no sols so no x-ints

$D = \{x \mid x \neq 0\}$

Range: need Calculator

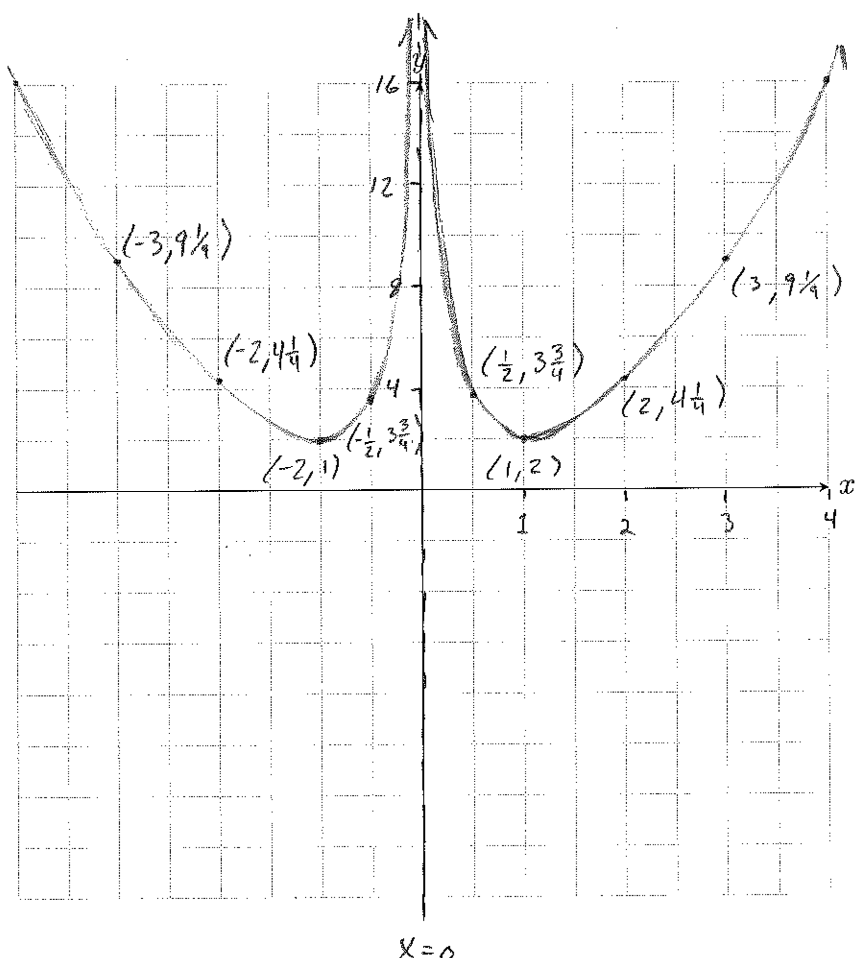
$R(-3) = R(3) = \frac{82}{9} = 9\frac{1}{9}$

$R(4) = R(-4) = \frac{257}{16} = 16\frac{1}{16}$

$R(2) = R(-2) = \frac{17}{4} = 4\frac{1}{4}$

$R(1) = R(-1) = 2$

$R(\frac{1}{2}) = R(-\frac{1}{2}) = \dfrac{\frac{1}{16}+1}{\frac{1}{4}} = \frac{17}{16} \cdot \frac{4}{1} = \frac{17}{4} = 3\frac{3}{4}$



$\leftarrow$ H.A. @ $y = \frac{3}{1} = 3$

9. Analyze and then graph the rational function $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$. $\leftarrow$ neither even nor odd

$R(x) = \dfrac{3x(x-1)}{(x+4)(x-3)}$

X-ints: $x=0$ or $x=1$
$(0,0)$ & $(1,0)$

$R(0) = 0$
y-int: $(0,0)$

V.A.s: $x=-4$, $x=3$
1st degree both

$D = \{x \mid x \neq -4, 3\}$

solve $R(x) = 3$: $3 = \dfrac{3x^2 - 3x}{x^2 + x - 12} \Rightarrow 3x^2 + 3x - 36 = 3x^2 - 3x$

$-36 = -6x$

$x = 6$

Does $R(x)$ cross the H.A.? $\longrightarrow$ Yes, @ $x=6$

$R(-6) = \dfrac{(-18)(-7)}{(-2)(-9)} = 14$

$R(-8) = \dfrac{(-24)(-9)}{(-4)(-12)} = 4.5$

$R(-7) = \dfrac{(-21)(-8)}{(-3)(-10)} = 5.6$

$R(-3) = \dfrac{(-9)(-4)}{(1)(-6)} = -6$

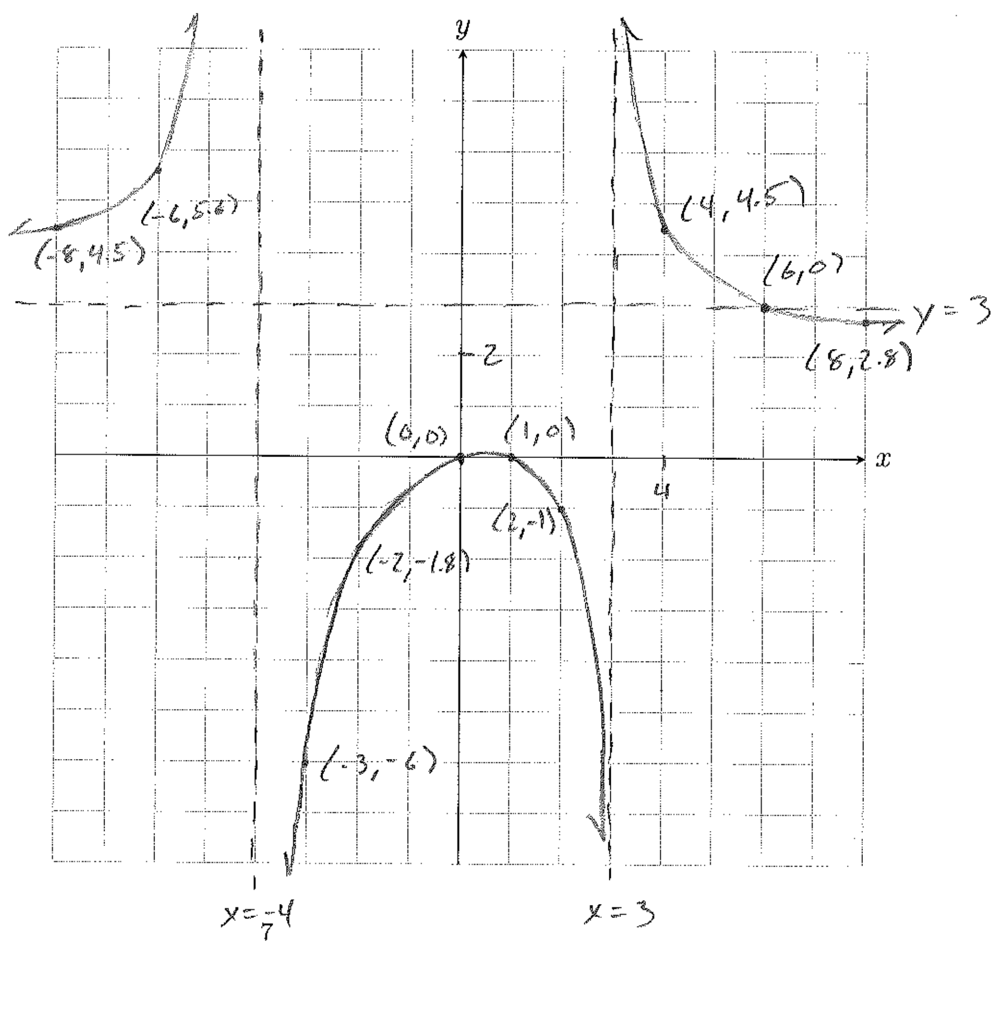
$R(-2) = \dfrac{(-6)(-3)}{(2)(-5)} = -9/5 = -1.8$

$R(2) = \dfrac{6}{-6} = -1$

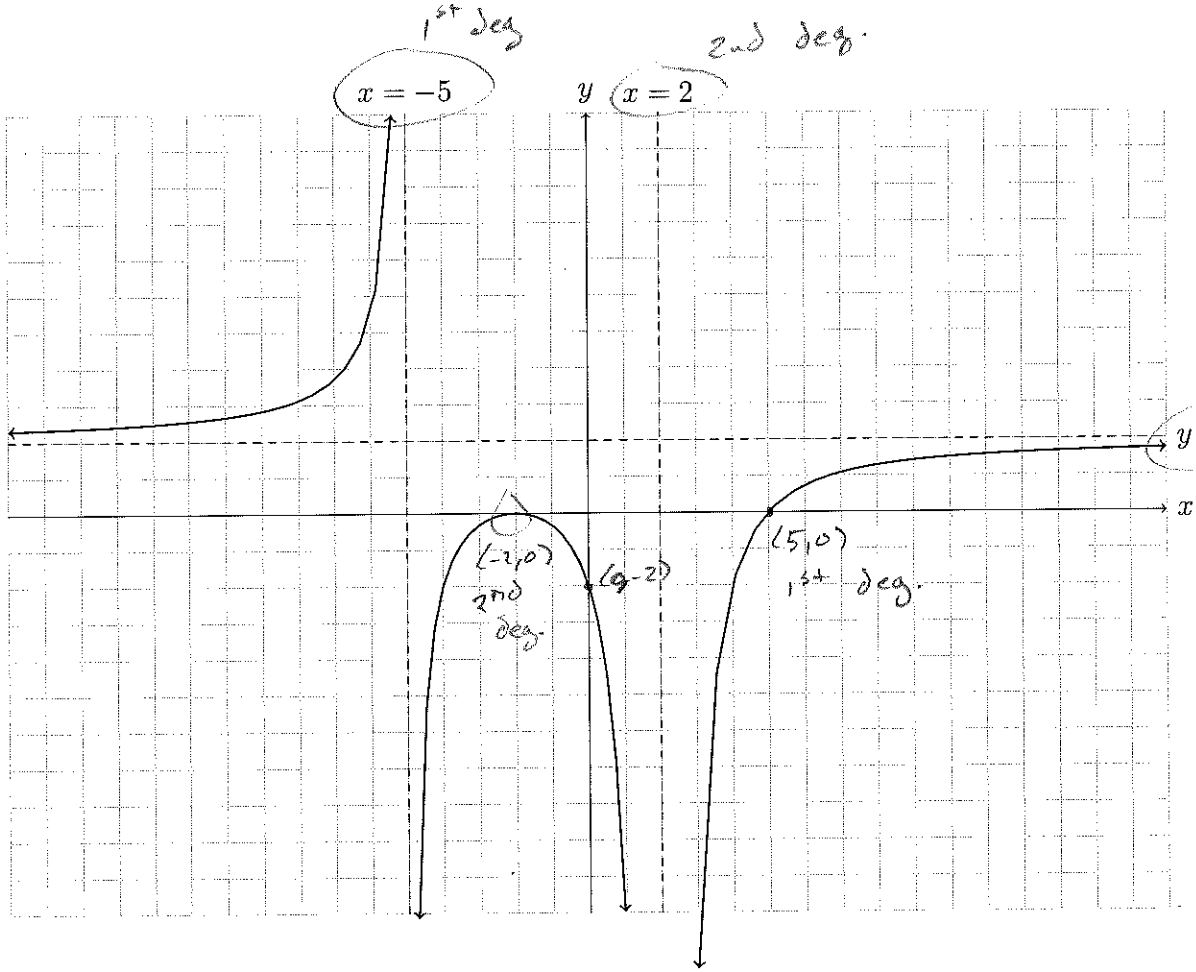
$R(4) = \dfrac{(12)(3)}{8} = 4.5$

$R(8) = \dfrac{(24)(7)}{(12)(5)} = \dfrac{14}{5} = 2.8$

Range: Need Calculator



10. Find a rational function that might have the following graph.



ratio of 1st terms

numerator & den have same degree

$$R(x) = \frac{2(x+2)^2(x-5)}{(x+5)(x-2)^2}$$

Test: $R(0) = \frac{2(2)^2(-5)}{(5)(-2)^2} = -2$ ✓