Solutions

Math 252 In Class Worksheet 1

Possible Limit Situations:

Horizontal Asymptote:

ex 1. $\lim_{x\to\infty} \frac{x}{x^2-x+1} = 0$

ex 2. $\lim_{x\to-\infty} \frac{3x^3-5x-2}{-5x^3+4} = -\frac{3}{5}$

both by multiplying by $\frac{1}{x^n}$ top and bottom where n is the degree of the denominator.

Composite Function:

ex 1. $\lim_{\theta\to\left(\frac{\pi}{2}\right)^-} \ln\left(\frac{1}{\tan(\theta)}\right) = -\infty$

since $\lim_{\theta\to\left(\frac{\pi}{2}\right)^-} \tan(\theta) = \infty$ and

$\lim_{\tan(\theta)\to\infty} \frac{1}{\tan(\theta)} = 0^+$ so

$\lim_{\frac{1}{\tan(\theta)}\to 0^+} \ln\left(\frac{1}{\tan(\theta)}\right) = -\infty.$

Vertical Asymptote:

ex 1. $\lim_{x\to 3} \frac{x-2}{(x-3)^2} = \infty$ by evaluating signs of the left and right limits.

Hole:

ex 1. $\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6} = -4$ by cancellation.

Continuity:

ex 1. $\lim_{\theta\to\frac{\pi}{2}} e^{\cos(\theta)} = 1$

by substitution since $e^{\cos(\theta)}$ is a continuous function.

There are also many **indeterminate** $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$ forms which may not fall into the hole or vertical asymptote situation but may still be evaluated. In order to do so we introduce:

l'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

$$OR \qquad \lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

In other words you have an **indeterminate** form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

so long as the limit on the right (with the derivatives) exists or is $\pm\infty$.

1. Let $f(x) = x^2 + x - 2$ and $g(x) = x^2 - x$. First show that $\lim_{x \to 1} \frac{f(x)}{g(x)}$ does in fact equal $\lim_{x \to 1} \frac{f'(x)}{g'(x)}$. Then use the limit definition of a derivative to show that they must equal each other.

$$\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{(x+2)(x-1)}{x(x-1)} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{2x+1}{2x-1} = \frac{3}{1} = 3 \checkmark$$

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \frac{\lim_{x \to 1} \frac{f(x)-f(1)}{x-1}}{\lim_{x \to 1} \frac{g(x)-g(1)}{x-1}} = \lim_{x \to 1} \frac{f(x)-f(1)}{g(x)-g(1)} = \lim_{x \to 1} \frac{f(x)}{g(x)}$$

Since $f(1) = g(1) = 0$

2. Determine the following limits using l'Hospital's Rule:

a. $\lim_{x \to 1} \frac{\ln(x)}{x-1} \overset{\frac{0}{0}}{=} \lim_{x \to 1} \frac{1/x}{1}$

$= 1$

b. $\lim_{x \to \infty} \frac{e^x}{x^2} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{e^x}{2x}$

$\overset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{e^x}{2} = \infty$

c. $\lim_{x\to 0} \frac{\tan(x) - x}{x^3}$ $\frac{0}{0}$ $= \lim_{x\to 0} \frac{\sec^2(x) - 1}{3x^2}$

$\frac{0}{0}$ $= \lim_{x\to 0} \frac{2\sec(x)\cdot\sec(x)\tan(x)}{6x}$

$= \lim_{x\to 0} \frac{2\sin x}{6x\cos^3 x}$

$\frac{0}{0}$ $= \lim_{x\to 0} \frac{2\cos x}{6\cos^3 x - 18x\cos^2(x)\sin(x)}$

$= 2/6 = 1/3$

d. $\lim_{x\to \pi^-} \frac{\sin(x)}{1 - \cos(x)}$

$= \frac{0}{1-(-1)} = 0$

e. $\lim_{x\to \left(\frac{\pi}{2}\right)^-} (\sec(x) - \tan(x))$ $\infty - \infty$ $= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \left(\frac{1-\sin(x)}{\cos(x)}\right)$

$\frac{0}{0}$ $= \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{-\cos(x)}{-\sin(x)}$

$= 0$

f. $\lim_{x\to 0^+} (1 + \sin(4x))^{\cot(x)}$

$= \lim_{x\to 0^+} e^{\cot(x)\ln(1+\sin(4x))}$

$\infty\cdot 0$ $= e^{\lim_{x\to 0^+} \frac{\cos(x)\ln(1+\sin(4x))}{\sin(x)}}$ $\frac{0}{0}$

$= e^{\lim_{x\to 0^+} \frac{-\sin(x)\ln(1+\sin(4x)) + \cos(x)\cdot\frac{4\cos(4x)}{1+\sin(4x)}}{\cos(x)}}$

$= e^4$

Practice Problems:

1. Evaluate the following limits.

a. $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$

$= \lim_{x\to\infty} \frac{3}{x^{1/3}}$

$= 0$

d. $\lim_{x\to 1} \frac{x^a - ax + a - 1}{(x-1)^2} \overset{0/0}{=} \lim_{x\to 1} \frac{ax^{a-1} - a}{2(x-1)}$

$\overset{0/0}{=} \lim_{x\to 1} \frac{a(a-1)x^{a-2}}{2}$

$= \frac{a(a-1)}{2}$

b. $\lim_{t\to 0} \frac{e^{3t}-1}{t} \overset{0/0}{=} \lim_{t\to 0} \frac{3e^{3t}}{1}$

$= 3$

e. $\lim_{x\to 0^+} (\sin(x)\ln(x)) \overset{0\cdot(-\infty)}{=} \lim_{x\to 0^+} \frac{\sin(x)}{(\ln(x))^{-1}}$

$\overset{0/0}{=} \lim_{x\to 0^+} \frac{\cos(x)}{-1\cdot(\ln(x))^{-2}\cdot\frac{1}{x}}$

$= \lim_{x\to 0^+} \frac{\cos(x)(\ln(x))^2}{-1/x}$

$= \lim_{x\to 0^+} \frac{\ln x}{(\sin(x))^{-1}}$

$\overset{\infty/\infty}{=} \lim_{x\to 0^+} \frac{1/x}{-(\sin(x))^{-2}\cos(x)}$

$= \lim_{x\to 0^+} -\frac{\sin^2(x)}{x\cos(x)}$

$\overset{0/0}{=} \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{\cos(x) - x\sin(x)}$

$= 0$

c. $\lim_{u\to\infty} \frac{e^{u/10}}{u^3} \overset{\infty/\infty}{=} \lim_{u\to\infty} \frac{\frac{1}{10}e^{u/10}}{3u^2}$

$\overset{\infty/\infty}{=} \lim_{u\to\infty} \frac{\frac{1}{100}e^{u/10}}{6u}$

$\overset{\infty/\infty}{=} \lim_{u\to\infty} \frac{\frac{1}{1000}e^{u/10}}{6}$

$= \infty$

f. $\lim_{x\to 0}(1-2x)^{1/x} = e^{\lim_{x\to 0} \frac{1}{x}\ln(1-2x)}$

$\overset{0/0}{=} e^{\lim_{x\to 0} \frac{-2/(1-2x)}{1}}$

$= e^{-2}$

2. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into accound, is

$$v = \frac{mg}{c}\left(1 - e^{-ct/m}\right)$$

where g is the acceleration due to gravity and c is a positive constant.

a. Calculate $\lim_{t\to\infty} v$. What is the meaning of this limit?

$$\lim_{t\to\infty} v = \frac{mg}{c} \lim_{t\to\infty} \left(1 - e^{-ct/m}\right)$$

$$= \frac{mg}{c}(1-0)$$

$$= mg/c$$

This means that the velocity tops out at $\frac{mg}{c}$ meters per second.

(Here, we assumed g is in meters/s^2)

b. For fixed t, use l'Hospital's Rule to calculate $\lim_{c\to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

$$\lim_{c\to 0^+} v = mg \lim_{c\to 0^+} \frac{1 - \frac{1}{e^{ct/m}}}{c}$$

$$\overset{0/0}{=} mg \lim_{c\to 0^+} \frac{\frac{t}{m} e^{-ct/m}}{1}$$

$$= gt$$

Here we may conclude that c must be an air resistance factor since as we take it to zero the velocity approaches linear growth as determined by gravity times time.