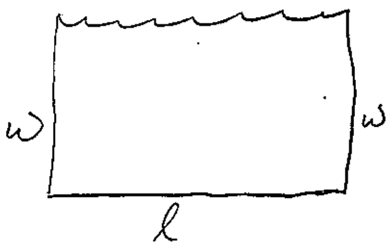
Math 252 [redacted] WS 2, Optimization (4.6) Solutions

1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?




Fencing = 2400 ft

Find w & l so that Area is maximized.

Let w = width in ft & l = length in ft

$2w + l = 2400$ ← constraint equation

$l = 2400 - 2w$ → Use in maximize equation to eliminate a variable.

Maximize $A = l \cdot w$ ← need equation with dimensions since being asked for & area since that's what's maximized.

$A(w) = (2400 - 2w) \cdot w$

$A(w) = -2w^2 + 2400w$

$D(A) = [0, 1200]$ since 2400 ft of fencing.

$\frac{dA}{dw} = -4w + 2400$ ← Use calculus to find critical numbers.

$0 = -4w + 2400$

$\Rightarrow w = 600$

Then check endpoints & critical outputs:

$A(0) = 0$

$A(1200) = 0$

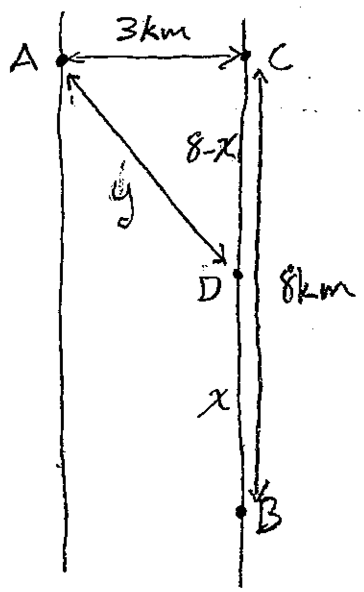
$A(600) = 720{,}000$ is the absolute max.

| | | | |
|---|---|---|---|
| $A'$ | + | | − |
| $A$ | ↗ | max | ↘ |
| | | 600 | |

So we have a local max when $w = 600$

The dimensions of the field that has the largest area with the constraint of a rectangular field fenced with 2400 ft along a river is 600 ft by 1200 ft.

2. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point $B$, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point $C$ and then run down to $B$, or he could row directly to $B$, or he could row to some point $D$ between $C$ and $B$ and then run to $B$. If he can row 6 km/hr and run 8 km/hr, where should he land to reach $B$ as soon as possible? (Assume the speed of the water is negligible compared to the speed at which he rows.)




Let x be the distance between B & D in km. & y the distance between A + D in km.

Note $\frac{dy}{dt} = 6$ km/hr & $\frac{dx}{dt} = 8$ km/hr.

Minimize T = total time in hours.

$$T = \frac{y}{6} + \frac{x}{8}$$

constraint: $y = \sqrt{9 + (8-x)^2}$

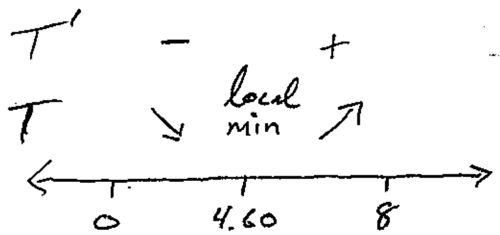
$$T(x) = \frac{1}{6}\sqrt{9+(8-x)^2} + \frac{1}{8}x \qquad \text{Dom}(T) = [0,8]$$

$$T'(x) = \frac{1}{12}(9+(8-x)^2)^{-1/2}(2(8-x)(-1)) + \frac{1}{8}$$

$$0 = \frac{x-8}{6\sqrt{9+(8-x)^2}} + \frac{1}{8} \Rightarrow x = 8 \pm \frac{9\sqrt{7}}{7}$$

$\approx 11.40$, $4.60$

↑ not in domain



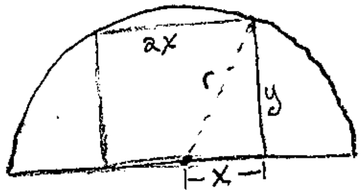

$T(0) \approx 1.42$

$T(4.60) \approx 1.33$

$T(8) = 1.5$

It would take about 1.42 hrs if he rowed directly to B, 1.5 hrs if he rowed directly to C then ran all the way from C to B, & the minimum time of about 1.33 hrs occurs if he rows to a location about 4.60 km north of B + runs the rest of the way.

3. Find the ~~area of the largest rectangle~~ rectangle of largest area that can be inscribed in a semi-circle of radius $r$.




Let $x$ be half the length of the rectangle & $y$ be the width as shown.

Maximize $A$ = area of rectangle

$A = 2xy$

Constraint:

$x^2 + y^2 = r^2$ ← constant

$y = \sqrt{r^2 - x^2}$

$A(x) = 2x\sqrt{r^2 - x^2}$ $\quad$ $Dom(A) = [0, r]$

$A'(x) = 2\sqrt{r^2 - x^2} + \dfrac{-2x^2}{\sqrt{r^2 - x^2}}$

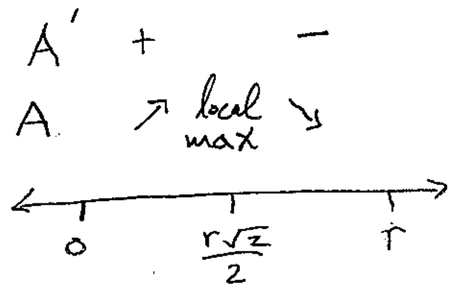
$-2\sqrt{r^2 - x^2} = \dfrac{-2x^2}{\sqrt{r^2 - x^2}}$

$-2(r^2 - x^2) = -2x^2$

$r^2 = 2x^2$

$x = \dfrac{r\sqrt{2}}{2}$

$\Rightarrow y = \dfrac{r\sqrt{2}}{2}$




$A(0) = 0$

$A(r) = 0$

$A\left(\frac{r\sqrt{2}}{2}\right) = r^2$

The area is maximized when the length is $r\sqrt{2}$ & the width is $\frac{r\sqrt{2}}{2}$. The maximum area is $r^2$.

**Practice Problems:**

4. A cylindrical can is to be made to hold 1 Liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

5. A ladder is being carried down a hallway 2 meters wide. At the end of the hall there is a right angled turn into a narrower hallway 1.5 meters wide. What is the length of the longest ladder that can be carried *horizontally* around the corner?

6. Find the area of the largest rectangle that can be inscribed in the ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Strategy for success when dealing with Optimization Story Problems, i.e. "Holy crap, and I thought related rates problems were hard! Problems."

a. Read the problem carefully then draw a picture of the situation.

   i. Use variables to label any lengths or angles that the problem suggests may be important.

   ii. Define those variables specifically including units.

b. State specifically any pertinent information given in the problem.

c. Assign a variable to the thing you are trying to minimize or maximize. Include units.

d. Write an equation which gives the thing you are trying to maximize or minimize.

e. Determine your constraint equation. This is often the most difficult part of this situation! Take your time and think critically about the situation. What do you know, what all is changing? One thing that can really help is understanding that the next step is to use the constraint equation to replace one or more variables in the equation we want to optimize found in the previous step.

f. Isolate an appropriate variable in the constraint equation and then substitute into the optimization equation to make the thing we are trying to optimize a function of a *single variable.*

g. State the domain of this function.

h. Take the derivative of the function and set it equal to zero to find the critical numbers.

i. Set up a number line and use the first derivative test to determine if the critical number is a local min or max.

j. Determine the absolute min or max (depending on what you're trying to do) for the interval in question. Plug the critical number back into the optimization function to obtain this value.

k. State a conclusion which states what the maximum or minimum value is and the values of the variables which allow for this optimized situation all in context of the problem.