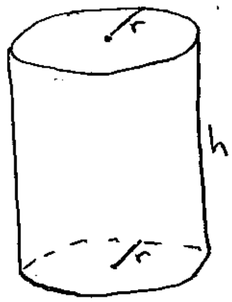
Practice Problems:

4. A cylindrical can is to be made to hold 1 Liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.



$C = 2\pi r$

Let r & h be the radius & height of the can in cm respectively. Let C be the circumference in cm.

Constraint

$V = \pi r^2 h = 1$ Liter $= 1000\,cm^3$

Note!

$\Rightarrow h = \dfrac{1000}{\pi r^2}$

Minimize Surface Area $= A = 2\pi r h + 2\pi r^2$

$$A(r) = 2\pi r \cdot \frac{1000}{\pi r^2} + 2\pi r^2 = 2000 r^{-1} + 2\pi r^2$$

$Dom(A) = (0, \infty)$ $\quad A'(r) = -2000 r^{-2} + 4\pi r$

$A'(r) = 0$ when $2000 = 4\pi r^3$

$\Rightarrow \sqrt[3]{\frac{500}{\pi}} = r$

$r = \dfrac{5\sqrt[3]{4\pi^2}}{\pi} \approx 5.42\,cm$

A' $-$ $+$

A $\searrow$ local min $\nearrow$

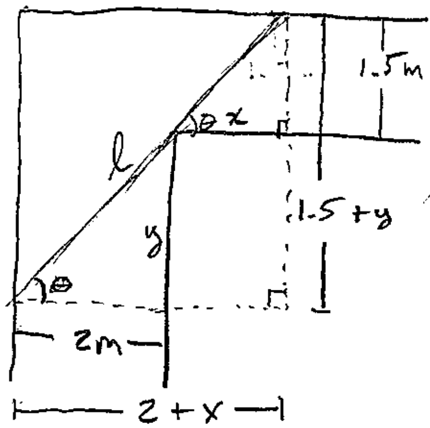
0 $\quad \dfrac{5\sqrt[3]{4\pi^2}}{\pi}$

$$A\left(\frac{5\sqrt[3]{4\pi^2}}{\pi}\right) = 300\sqrt[3]{2\pi} \approx 553.58\,cm^2$$

If $r = \dfrac{5\sqrt[3]{4\pi^2}}{\pi}$ then $h = \dfrac{10\sqrt[3]{4\pi^2}}{\pi} \approx 10.841\,cm$

The cost of the can is minimized when the radius is about 5.42 cm & the height is about 58.74 cm. This gives a minimal surface area of about $553.58\,cm^2$.

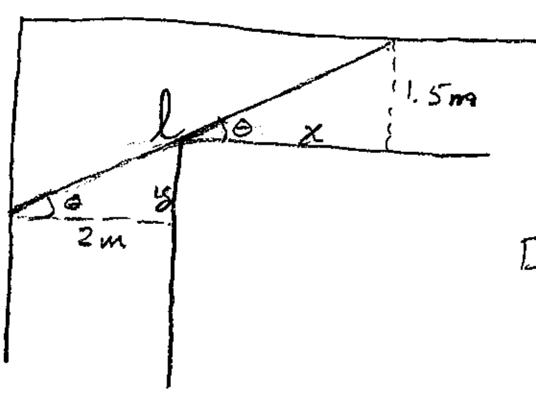
~~X~~ 5. A ladder is being carried down a hallway 2 meters wide. At the end of the hall there is a right angled turn into a narrower hallway 1.5 meters wide. What is the length of the longest ladder that can be carried *horizontally* around the corner?



Assume negligible width when the ladder is held horizontally but dangling so the rungs are vertical.

Let x, y, + l be the lengths shown in the picture in meters. Let θ be the angle shown. Note it's the same θ.

If we make the ladder as long as possible for different θ:



We see the longest possible ladder to take around the corner is the one we get when minimizing l with respect to theta.

Constraint: $y = 2\tan\theta$

+ $x = \dfrac{1.5}{\tan\theta}$

$$l = \sqrt{(2+x)^2 + (1.5+y)^2}$$

$$l(\theta) = \sqrt{\left(2+\frac{1.5}{\tan\theta}\right)^2 + (1.5+2\tan\theta)^2}$$

is minimized when

$$m(\theta) = \left(2+\frac{1.5}{\tan\theta}\right)^2 + (1.5+2\tan\theta)^2$$

is minimized

$$\mathrm{Dom}(l) = \left(0, \frac{\pi}{2}\right)$$

$$m'(\theta) = -\frac{(3\cos\theta + 4\sin\theta)(3\cos^3\theta - 4\sin^3\theta)}{2\sin^3\theta\cos^3\theta}$$

$m'(\theta) = 0$ when $\theta \approx 0.74$ radians

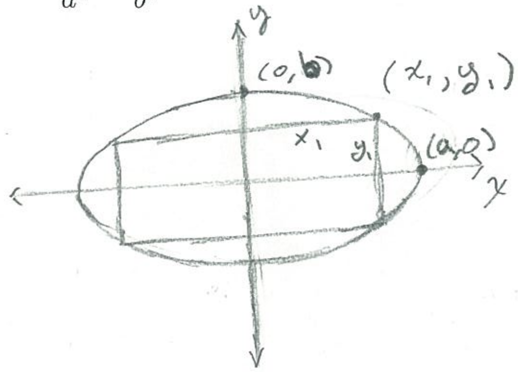
Both via CAS.

m'		$-$	$+$
m		$\searrow$ loc. min $\nearrow$	
	0	0.74	

$l(0.74) \approx 4.93$ meters

The longest ladder to round this corner in this manner is about 4.93 meters long.

~~5.~~ 6. Find the area of the largest rectangle that can be inscribed in the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Let x_1 be half the length & y_1 be half the width. Let the area of one quarter of the whole rectangle be $A = x_1 y_1$.

Note the area of the whole rectangle is $4A$ & is maximized when A is maximized.

Constraint:

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

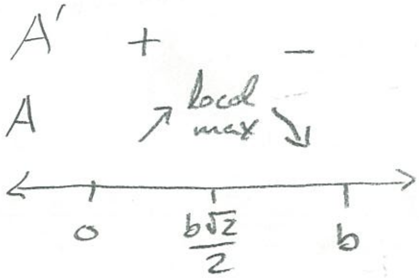
$$\Rightarrow \quad x_1 = \sqrt{a^2 - \frac{a^2}{b^2} y_1^2}$$

$$= a\sqrt{1 - \frac{1}{b^2} y_1^2}$$

$$A(y_1) = a\sqrt{1 - \frac{1}{b^2} y_1^2} \cdot y_1 \qquad \mathrm{Dom}(A) = [0, b]$$

$$A'(y_1) = \frac{a}{b}\sqrt{b^2 - y_1^2} - \frac{a y_1^2}{b\sqrt{b^2 - y_1^2}} \quad \text{via CAS}$$

$$A'(y_1) = 0 \quad \text{when } y_1 = \frac{b\sqrt{2}}{2} \quad \text{via CAS}$$



$$A(0) = A(b) = 0$$

$$A\left(\frac{b\sqrt{2}}{2}\right) = \frac{ab}{2}$$

$\Rightarrow$ The maximum area of a rectangle inscribed in an ellipse with horizontal length $2a$ & vertical width $2b$ is $4A = 2ab$ which occurs when (x_1, y_1) is chosen to be

$$\left(\frac{a\sqrt{2}}{2}, \frac{b\sqrt{2}}{2}\right).$$