Math 95 WS 3 Linear Equations and Inequalities   Name: ______________________________

§3.1, 3.3, 3.4, 3.5

We now turn to solving linear equations. I will assume that everyone in this class has some experience solving linear equations and will begin with a review of a technique for solving linear equations which involve fractions.

Suppose I had an equation such as

$$\tfrac{1}{3}(2z-3)-\tfrac{1}{2}z=-2$$

In this situation, or any situations involving an equation where you are solving for a variable and fractions are involved, a very helpful technique is to begin by multiplying both sides of the equations by the LCD. Whatch what happens when we utilize this technique in the equation given above:

$$\begin{aligned}
\frac{1}{3}(2z-3)-\frac{1}{2}z &= -2\\
6\left(\frac{1}{3}(2z-3)-\frac{1}{2}z\right) &= 6(-2)\\
6\cdot\frac{1}{3}(2z-3)-6\cdot\frac{1}{2}z &= -12\\
2(2z-3)-3z &= -12\\
4z-6-3z &= -12\\
z-6 &= -12\\
z &= -6
\end{aligned}$$

I would like to emphasize the logic in this procedure. A question I may ask on an exam is, "why is it that multiplying both sides of an equation involving fractions by the LCD will inevitably clear all the fractions?" The answer follows directly from the definition of the LCD. The Least Common Denominator is the Least Common Multiple of each denominator. This means that each denominator *goes into* the LCD and hence will be cancelled out by the LCD leaving no fractions. Yay! Try these on for practice:

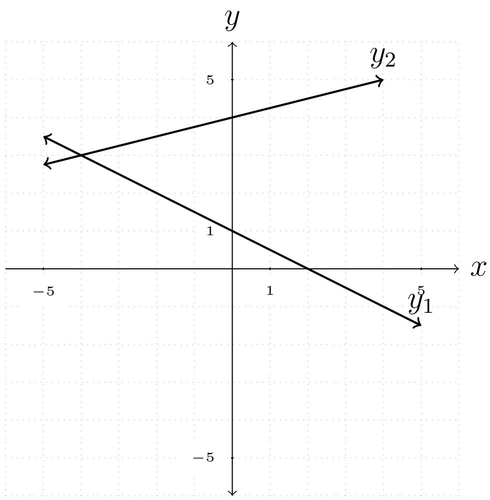
1. $\frac{1}{3}x-5=\frac{1}{10}x+2$

2. $\frac{1}{4}(x-2)+\frac{3}{10}x=\frac{5}{6}x$

For the purpose of emphasizing the fact that graphs and symbolic representations are two different ways of expressing the same information, let's look at how we may solve a linear equations graphically. Suppose we had the equation

$$1 - \frac{1}{2}x = \frac{1}{4}x + 4.$$

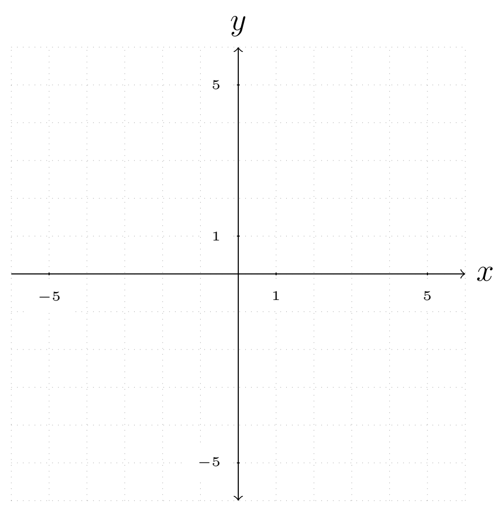
I could set up a linear function for each *side* of this equation in the following manner. I will let the left hand side of the above equation be $y_1(x) = 1 - \frac{1}{2}x$ and the right hand side be $y_2(x) = \frac{1}{4}x + 4$. We can now graph each of these on the same set of axes to find the solution to the system of equations.



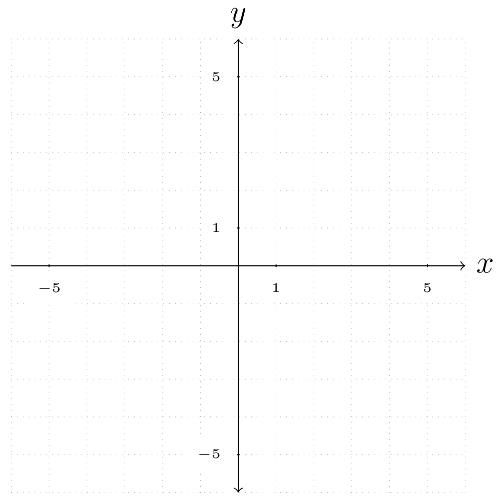
From this we can see that the two functions share the ordered pair (-4,3). That is, (-4,3) is a solution to the system of equations given by the functions y_1 and y_2.
Now, the initial equation we were given wasn't looking for a y value. We introduced y values into play when we created the functions, however the initial question is simply going to be the x value which allows $y_1(x)$ to equal $y_2(x)$. Thus the solution in this instance is just $x = -4$. Or, in set notation, $\{-4\}$.

Try solving the following equations by graphing on the given coordinate plane.

3. $\frac{3}{4}x - 5 = -\frac{1}{2}x + 3$



4. $3 = 4 - \frac{1}{2}(4 + x)$



What happens if I have the linear equation $2(z - 1) + z = 3z + 2$? Go ahead and solve for z.

What do we know is going on when this happens?

What do you think would happen if we tried to find the solution to this equation by graphing as we did above?

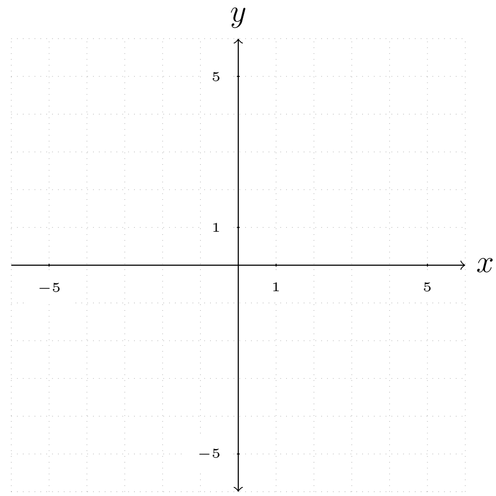
Now try solving $2x + 3(x - 7) = 4x - 21 + x$. What is going on in this situation and what would happen if we tried finding the solution(s) to this equation graphically?

Quick review: What is the standard form of the equation for a line?

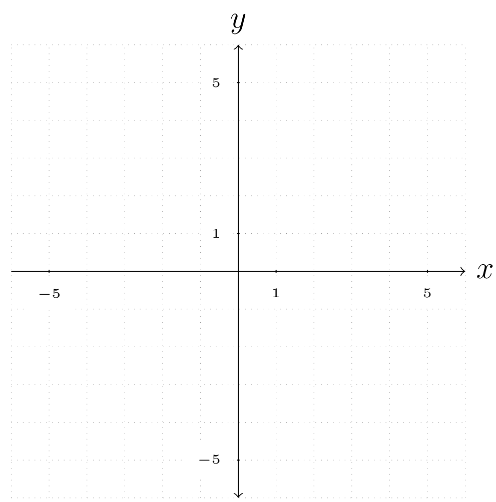
How do we find the x-intercepts of a line when given an equation for a line or linear function?

How do we find the y-intercepts of a line when given an equation for a line or linear function?

5. Find the x and y intercepts for $3x - 2y = 6$ and then use these points to graph the line.



6. Find the x and y intercepts for the function $f(x) = 2x - 8$ and then use these points to graph the line.



7. In 1992 private industry reported 8.3 injuries per 100 full-time workers. By 2002 this number decreased to 5.4.

   a. Find a linear function that models the data.

   b. Estimate the year when the reported injuries equaled 6 per 100 full-time workers.

So, what does an equal sign do? Or, what is it used for?

However, when we usually are comparing two quantities, they will not have equivalent values. Usually one will be larger than the other. Oftentimes, we will want to indicate that a particular expression may be greater than OR equal to another expression or value. When this is the case we use inequalities to make this indication. The inequalitiy symbols are $>$, $<$, $\geq$ and $\leq$. Why do we call them *in*equalities?

Remember, if I am solving an equation, I am looking for a value for the variable(s) which will make the equal sign true. If I'm solving an *inequality* I am looking for the value or values that the variable(s) may take on which will make the inequality symbol true. For example, in the equation $2x - 12 = -4x$ I would say that $x = 2$ since plugging 2 in for x in the equation will result in a true statement. The difference between this and solving $2x - 12 > -4x$ is that in this inequality we find that $x = 3$, $x = 10$ and an infinite number of other possibilites will also make the $>$ symbol true. To solve this inequality, we go about it *almost* as if we were solving an equation, with the only caveat coming when you need to divide or multiply both sides by a negative value. In other words, we treat the $>$ as if it were an equal sign unless dividing or multiplying by a negative number:

$$\begin{aligned} 2x - 12 &> -4x \\ -2x \quad &\quad -2x \\ -12 &> -6x \\ \overline{-6} \quad &\quad \overline{-6} \\ 2 &< x \end{aligned}$$

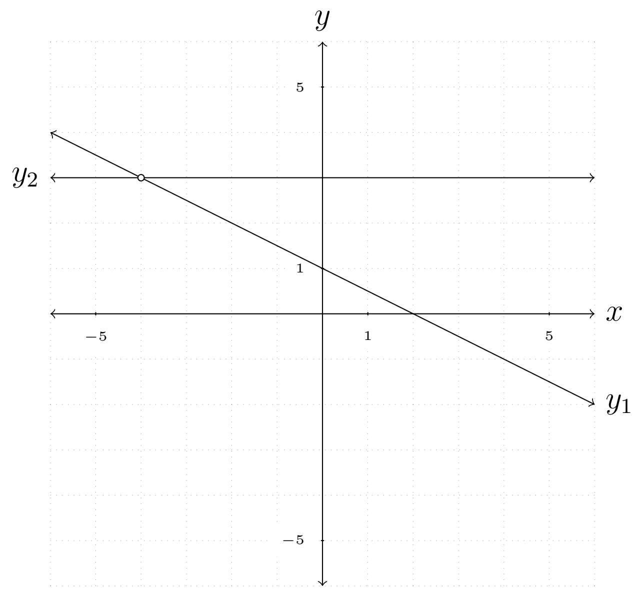
So our solution set is $\{x | x > 3\}$.
Have some practice! Solve the following inequalities.

8. $2x - 1 > 4$

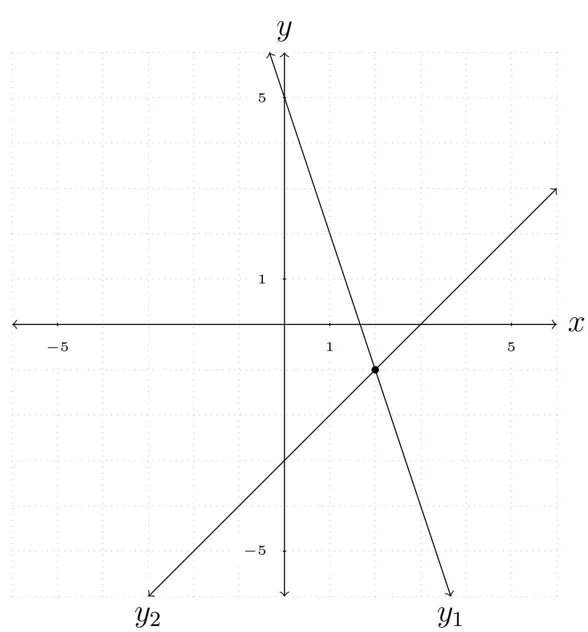
9. $\frac{1}{2}(z - 3) - (2 - z) \leq 1$

10. $\frac{2t-3}{5} \geq \frac{t+1}{3} + 3t$

Let's now see what the solutions to an inequality look like graphically. Let me give two different examples.
Given the inequality $-\frac{1}{2}x + 1 > 3$ we define functions for each side and then graph each function. Let $y_1(x) = -\frac{1}{2}x + 1$ and $y_2(x) = 3$. We use the slope and y-intercept to graph y_1 and remember y_2 graphs a horizontal line.

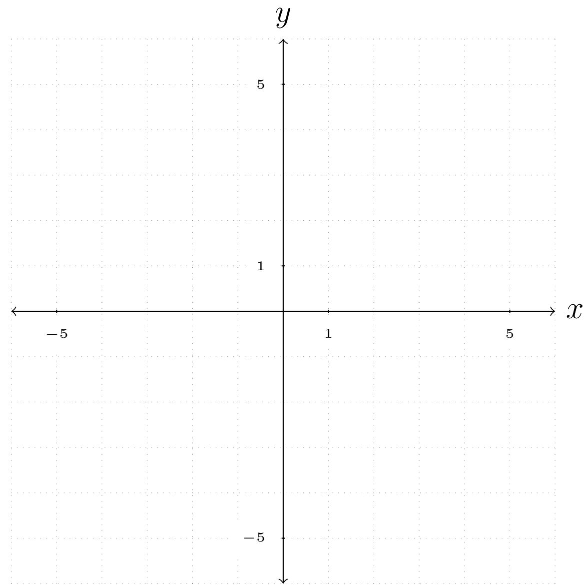


Now, the inequality is essentially asking, "when" is y_1 greater than y_2. When I use the word "when," I mean "for what x-values is this true?" In this case we see that y_1 is greater than y_2 when x is *less* than -4. That is y_1 is *above* y_2 when $x < -4$. Not that we are using *strictly* less than -4 since $y_1 = y_2$ when $x = -4$. We would write our answer as $\{x|x < -4\}$.
Let's do one more.
Given the inequality $5 - 3x \leq x - 3$, we define the functions $y_1(x) = 5 - 3x$ and $y_2(x) = x - 3$ and then use the slope and y-intercept to graph each.



Here we see that y_1 is less than *or* equal to y_2 when x is greater than or equal to 2. This is because y_1 goes below y_2 when x hits 2. Here we write our solution as $\{x|x \geq 2\}$.

11. Find the solution to $\frac{2}{3}x - 1 < -\frac{1}{3}x + 2$ using the graphing method described above. Make sure and define your functions first and then label each graph.



This whole idea of solving inequalities is usually used when discussing the domain of a function. However it can be useful in applications as well. For example, if ground-level temperature is 90°F, the temperature T above Earth's surface is modeled by $T(x) = 90 - 19x$, where x is the altitude in miles. Suppose that the clouds will form only if the temperature is 53°F or colder. Now, in the past we would simply set $T(x)$ equal to 53 and then solve for x: $53 = 90 - 19x$ which, after a little algebra results in $x = \frac{37}{19} \approx 1.95$ miles. But we wouldn't conclude that the clouds form only if the altitude is approximately 1.95 miles, they form when the altitude is *greater* than or equal to 1.95 miles, so our solution should be $x \geq \frac{37}{19}$. Instead of making this logical jump after doing work, we set up the problem from the beginning in a manner that takes care of this issue. We say that the clouds form only if $T(x) \leq 53$°F so we set up the inequality

$$\begin{aligned} 90 - 19x &\leq 53 \\ -90 \qquad &\quad - 90 \\ \hline -19x &\leq -37 \\ \overline{-19} \qquad &\quad \overline{-19} \\ x &\geq \frac{37}{19} \end{aligned}$$

Thus rendering our conclusion that the clouds form when the altitude is greater than or equal to approximately 1.95 miles.

We next look at what happens when we deal with what we call *compound* inequalities. Before jumping right in, let's discuss *truth tables*, one of my favorite things. Truth tables are simply a visual representation of how logical devices work. We will look at the truth tables for only to devices, the OR and the AND. Before looking at the tables, let me scramble your brain.

Attempt to define the word OR without using the word or, or any of its synonyms or antonyms.

You'll find similar difficulty when trying to define AND. Logicians define these words using the following tables:

C1	C2		OR
T	T		T
T	F		T
F	T		T
F	F		F

C1	C2		AND
T	T		T
T	F		F
F	T		F
F	F		F

Here, C1 and C2 are conditional statements while the T and F values below them represent the truth or falsity of the conditional. The T or F in the third column represents the truth or falsity of the resultant OR or AND statement. Examples:

I am from Mars OR we are in Oregon.
Here, our conditiionals are C1: I am from Mars; C2: we are in Oregon. C1 is False while C2 is True. If we look at the truth table above, we see that in this situation the resultant sentence is a true statement. On the other hand:
I am from Mars AND we are in Oregon.
Here we have the same conditionals C1 and C2 with C1 False and C2 true. Now we look at the truth table for AND to find that the resultant statement is false.

Back to numbers. Suppose we make a statement that a certain situation is true only when $x < 9$ AND $x \geq -4$. This is called a *compound* inequality. Simply multiple inequality conditions. Let's look at a number line for this:

With an AND statement we needed the *intersection* of the first two graphs!
If I were to describe this set in simplest form I would write $\{x| - 4 \leq x < 9\}$. I may also use interval notation and write $[-4, 9)$. Interval notation is really just the line notation condensed.
How would we graph $x < 9$ OR $x \geq -4$?

Here, with an OR statement we needed the *union* of the first two graphs!

In this case I would describe the set like this: $\mathbb{R}$. If we want to use interval notation here we would write $(-\infty, \infty)$.

Suppose we have the compound inequality $x \leq 3$ OR $x > 5$. This would graph in the following way:

If we want to write out this set in set notation we write $\{x | x \leq 3 \text{ OR } x > 5\}$. Here there's no simplification of the conditions. If we want to write this in interval notation we would write $(-\infty, 3] \cup (5, \infty)$. The $\cup$ symbol denotes the word *union* and we read this as "the set of numbers from negative infinity to 3, including 3, unioned with the numbers from 5 to infinity." This is a description of which numbers satisfy the compound inequality given above.

Let's see how to deal with $x \leq 3$ AND $x > 5$. First let's write this in interval notation: $(-\infty, 3] \cap (5, \infty)$. The $\cap$ symbol denotes the word *intersection* and we read this as "the set of numbers from negative infinity to 3, including 3, intersected with the numbers from 5 to infinity." An intersection of two sets is the numbers that belong to BOTH sets. Let's graph this compound inequality:

Here we see that the sets don't overlap and therefore the solution set to this compound inequality is empty. In set notation we write or $\{\}$.

Let's have you try a few. Keep in mind that you will need to solve each inequality first. Then graph, write out the set notation and the interval notation for each compound inequality.

12. $2x + 4 > 8$ AND $5 - x < 9$

13. $x + 2 < -1$ OR $x + 2 > 1$

A three-part inequality looks like $-5 < z \leq 7$. It is actually just a compressed version of $-5 < z$ AND $z \leq 7$. If I have an unsolved three-part inequality, it looks something like this: $10 < 4t + 2 \leq 14$. Let's work this out and then graph, write the set notation and the interval notation:

I think you can get a couple of these without to much trouble:

14. $-3 \leq 3x \leq 6$

15. $-\frac{5}{2} < \frac{1-m}{2} < 4$

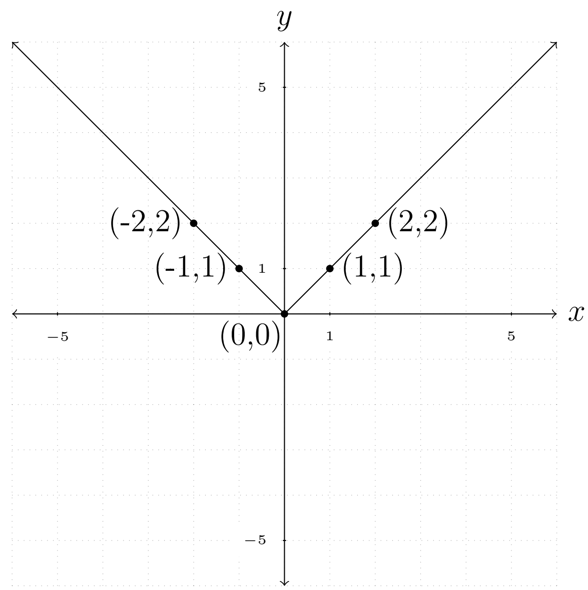
Great! Now let's look at what a solution to a three-part inequality looks like on a graph.

Tuition and fees at private colleges and universities from 1980 to 2000 can be modeled by $f(x) = 575(x - 1980) + 3600$. Use a graphing utility to estimate when the average tuition and fees ranged from \$8,200 to \$10,500.

We now move on to looking at solutions of absolute value equations and inequalities. To begin, let's dissect the absolute value function $f(x) = |x|$. Remember, this function takes any value and simply makes it positive. So, if I create a table which partially represents this function I may get something like:

x	$f(x) = \|x\|$
-2	2
-1	1
0	0
1	1
2	2

And if I graph $f(x)$ I get the following:



Now, to understand what the solutions to an absolute value *equation* are let's look at the following examples:

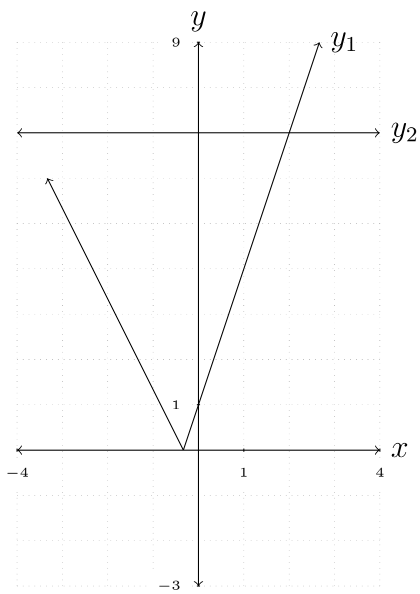
If I say that $|x| = 5$, you should be able to tell me that $x = 5$ or $x = -5$. I want you to think of this slightly different than this intuitive response. I want you to think $x = 5$ or $-x = 5$. See the subtle difference? We end up with the same result, but here we are thinking "whether the input is positive or negative, it results in the same value."

What if we want to solve $|3x + 1| = 7$? Here we employ the idea that everything inside the parenthesis must equal 7 or the *opposite* of everything in the parenthesis must equal 7. Thus we set up the following:

$$3x + 1 = 7 \qquad \text{or} \qquad -(3x + 1) = 7$$

If I were to use a graph to find the solutions to this equation we would set $y_1(x) = |3x+1|$ and $y_2(x) = 7$ and graph them both on the same set of axes. To graph $|3x+1|$ we will first fill out a table of values.

x	$y_1(x) = \|3x+1\|$
-3	8
-2	5
-1	2
$-\frac{1}{3}$	0
0	1
1	4
2	7



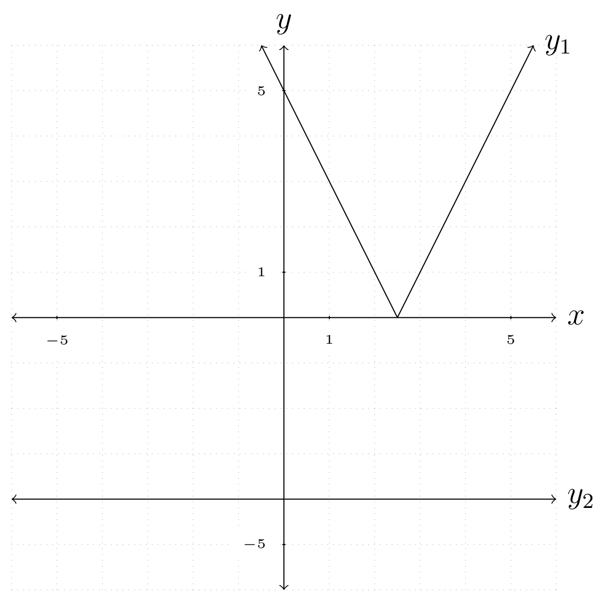
And then we see that the points of intersection of the two graphs are $\left(-\frac{8}{3}, 7\right)$ and (2,7). Again, we are only concerned with the x values in this situation so our solution set would be $\{-\frac{8}{3}, 2\}$. What if I ask what the solutions to $|-x+3| = 0$ are? Here we look at the entire inside of the absolute value and apply the same reasoning as above. That is, $-x+3=0$ or $-(-x+3)=0$. Now, in this case both situations result in $x=3$ as our solution, but that's okay, it's good to realize that this will happen but also good to pay attention to what the absolute value is defined to do.

Quick aside:

16. Does $|-x+3| = x+3$? Why or why not?

Now let's try and solve $|2x-5| = -4$. Here there are no solutions! Why not?

Here's the graphical representation of this situation where I'm letting $y_1(x) = |2x - 5|$ and $y_2(x) = -4$.



Sure enough, we see that the two graphs do NOT intersect so there are no solutions to the given equation.

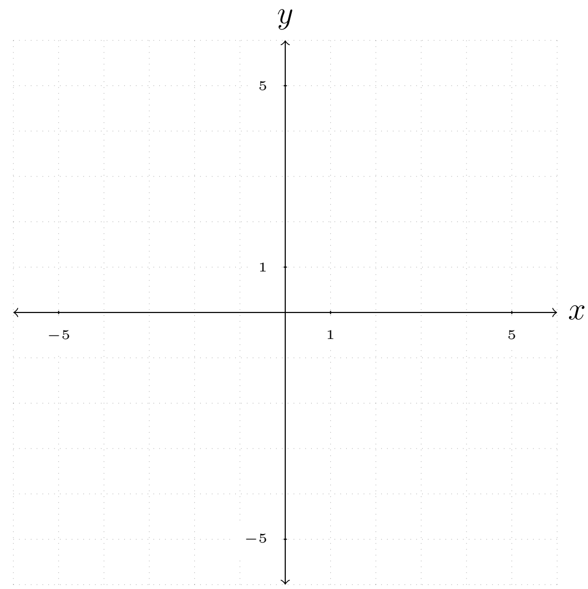
Finally, if I want to solve $|2x - 4| = |\frac{1}{2}x - \frac{1}{2}|$ for x we might think that each absolute value needs to be viewed as possibly positive and possibly negative. This would result in the 4 following situations:

$$2x - 4 = \tfrac{1}{2}x - \tfrac{1}{2} \quad \text{or} \quad -(2x-4) = \tfrac{1}{2}x - \tfrac{1}{2} \quad \text{or} \quad 2x - 4 = -\left(\tfrac{1}{2}x - \tfrac{1}{2}\right) \quad \text{or} \quad -(2x-4) = -\left(\tfrac{1}{2}x - \tfrac{1}{2}\right)$$

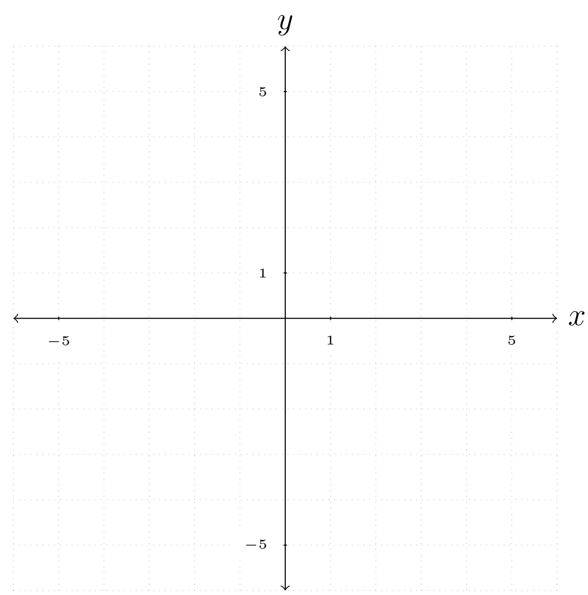
So we see that we only needed the first and second of the above two possibilities. The graphical solution with $y_1(x) = 2x - 4$ and $y_2(x) = |\frac{1}{2}x - \frac{1}{2}|$ can be found using our calculator as well. Let's give it a shot.

Here's some practice for you. Solve the following absolute value equations symbolically and graphically.

17. $|3x - 5| = 4$



18. $|-x + 3| = |2x - 3|$



Now, if we are dealing with inequalities instead of *e*qualities, then things get a bit more complicated. Suppose we have the absolute inequality $|2x-1|<3$. To find the soltuion to this we want to know *for what x values is this inequality true.* We can solve this symbolically just as we did with equations so long as we remember that the inequality symbol gets flipped if we divide by a negative. So, we'll set up

$$2x-1<3 \qquad \text{AND} \qquad -(2x-1)<3$$

It's important to remember that if the absolute value is less than the number then we set up an AND statement. We can see why when we look at the graphical solution. Go ahead and draw the graphs of both $y_1(x)=|2x-1|$ and $y_2(x)=3$.

If we look at $\left|\frac{2x-5}{3}\right|\geq 3$ then we set up the following compound inequality:

$$\frac{2x-5}{3}\geq 3 \qquad \text{OR} \qquad -\frac{2x-5}{3}\geq 3$$

Here we see that when the absolute value is *greater* than the number then we set up an OR statement. Let's look at the graphical solution to see why. Graph $y_1(x) = |\frac{2x-5}{3}|$ and $y_2(x) = 3$.

One last quick thing before you get practice is what happens when we have a situation like $|-x+3|+5 < 2$ or $|x-2|+3 > 1$. Graph each of these and we will discuss their solution sets.

Great! Here's your practice: Solve the following absolute value inequalities symbolically and graphically. Write your answers in set notation and interval notation.

19. $|frac3x - 22| < 2$

20. $|5x + 3| - 2 > -1$