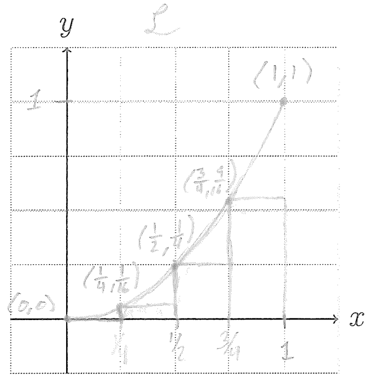
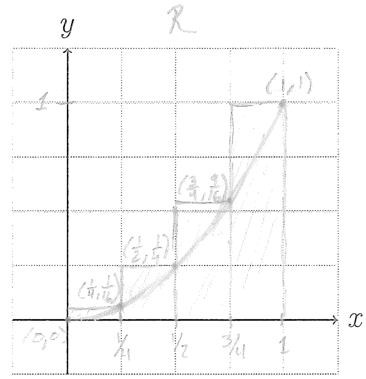
Solutions

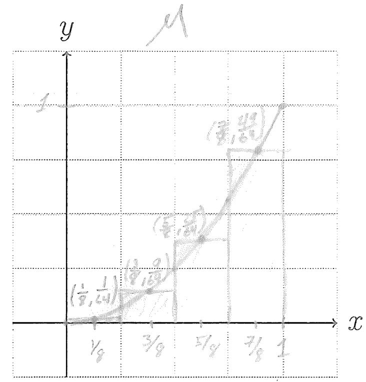
Math 252 WS 4 - Area Approximations

The Area Problem:

1. Approximate the area under $f(x) = x^2$ over the interval $[0, 1]$ with 4 rectangles using left endpoints, right endpoints, and midpoints.







$A \approx \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{9}{16}$

$= \frac{1}{4}\left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right)$

$= \frac{1}{4}\left(\frac{7}{8}\right) = \frac{7}{32}$

$A \approx \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{9}{16} + \frac{1}{4}\cdot 1$

$= \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right)$

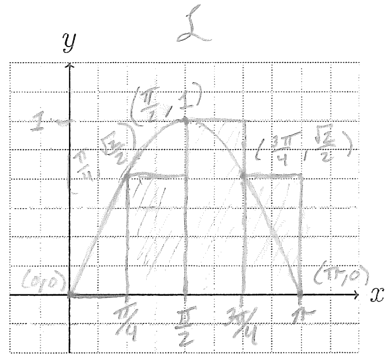
$= \frac{1}{4}\left(\frac{15}{8}\right) = \frac{15}{32}$

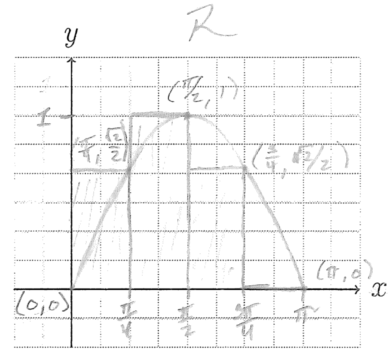
$A \approx \frac{1}{4}\cdot\frac{1}{64} + \frac{1}{4}\cdot\frac{9}{64} + \frac{1}{4}\cdot\frac{25}{64} + \frac{1}{4}\cdot\frac{49}{64}$

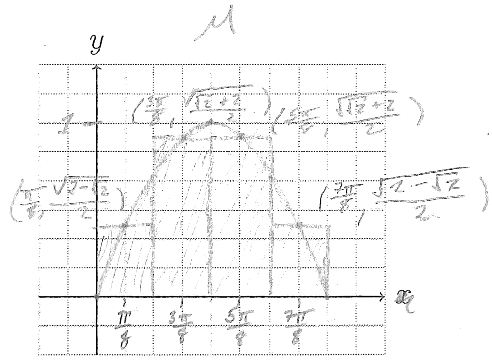
$= \frac{1}{4}\left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right)$

$= \frac{1}{4}\left(\frac{21}{16}\right) = \frac{21}{64}$

2. Approximate the area under $f(x) = \cancel{\cos(x)}$ $\sin(x)$ over the interval $[0, \pi]$ with 4 rectangles using left endpoints, right endpoints, and midpoints.







$A \approx \frac{\pi}{4}\cdot 0 + \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}$

$= \frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right)$

$= \frac{\pi(\sqrt{2}+1)}{4}$

$A \approx \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\pi}{4}\cdot 0$

$= \frac{\pi(\sqrt{2}+1)}{4}$

$A \approx \frac{\pi}{4}\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{\sqrt{2}+2}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}\right)$

$= \frac{\pi}{4}\left(\sqrt{2-\sqrt{2}} + \sqrt{\sqrt{2}+2}\right)$

3. Find an expression for the approximate area under $f(x) = x^2$ over the interval $[0, 1]$ using left endpoints with 8, 16, and n rectangles. Then use your CAS to take the limit as $n \to \infty$ to find the exact area.

$$A_8 = \frac{1}{8}\left(0^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{2}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{4}{8}\right)^2 + \left(\frac{5}{8}\right)^2 + \left(\frac{6}{8}\right)^2 + \left(\frac{7}{8}\right)^2\right) = \frac{1}{8}\sum_{i=0}^{7}\left(\frac{i}{8}\right)^2$$

$$= \frac{35}{128} \approx 0.2734$$

$$A_{16} = \frac{1}{16}\left(0^2 + \left(\frac{1}{16}\right)^2 + \left(\frac{2}{16}\right)^2 + \left(\frac{3}{16}\right)^2 + \left(\frac{4}{16}\right)^2 + \left(\frac{5}{16}\right)^2 + \left(\frac{6}{16}\right)^2 + \left(\frac{7}{16}\right)^2 + \left(\frac{8}{16}\right)^2 + \left(\frac{9}{16}\right)^2 + \left(\frac{10}{16}\right)^2 + \left(\frac{11}{16}\right)^2 + \left(\frac{12}{16}\right)^2 + \left(\frac{13}{16}\right)^2 + \left(\frac{14}{16}\right)^2 + \left(\frac{15}{16}\right)^2\right)$$

$$= \frac{1}{16}\sum_{i=0}^{15}\left(i/16\right)^2$$

$$= \frac{155}{512} \approx 0.3027$$

$$A_n = \underbrace{\frac{1-0}{n}}_{\Delta x = \frac{b-a}{n}}\left(0^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right) = \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=0}^{n-1} i^2$$

$$A = \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=0}^{n-1} i^2 = 1/3$$

4. Find an expression for the approximate area under $f(x) = \cancel{\cos(x)}$ sin over the interval $[0, \pi]$ using left endpoints with 8, 16, and n rectangles. Then use your CAS to take the limit as $n \to \infty$ to find the exact area.

$$A_8 = \frac{\pi}{8}\left(\sin(0) + \sin(\pi/8) + \sin\left(\frac{2\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) + \sin\left(\frac{4\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) + \sin\left(\frac{6\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right)\right)$$

$$= \frac{\pi}{8}\sum_{i=0}^{7}\sin(i\pi/8) \approx 1.974$$

$$A_{16} = \frac{\pi}{16}\left(\sin(0) + \sin(\pi/16) + \sin\left(\frac{2\pi}{16}\right) + \sin\left(\frac{3\pi}{16}\right) + \sin\left(\frac{4\pi}{16}\right) + \sin\left(\frac{5\pi}{16}\right) + \cdots + \sin\left(\frac{15\pi}{16}\right)\right)$$

$$= \frac{\pi}{16}\sum_{i=0}^{15}\sin\left(i\pi/16\right) \approx 1.9936$$

$$A_n = \frac{\pi - 0}{n}\left(\sin(0) + \sin(\pi/n) + \sin(2\pi/n) + \cdots + \sin\left(\frac{(n-1)\cdot\pi}{n}\right)\right)$$

$$= \frac{\pi}{n}\sum_{i=0}^{n-1}\sin(i\pi/n)$$

$$A = \lim_{n\to\infty}\frac{\pi}{n}\sum_{i=0}^{n-1}\sin\left(\frac{i\pi}{n}\right) = 2$$

5. Find an expression for the exact area under $f(x) = x^3 + 1$ over the interval $[0, 3]$ and then use a CAS to evaluate it.

$\Delta x = \frac{3}{n}$ $\quad x_i = 3i/n$ $\quad f(x_i) = \left(\frac{3i}{n}\right)^3 + 1$

$$A = \lim_{n\to\infty} \Delta x \sum_{i=0}^{n-1} f(x_i) = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)^3 + 1$$

$$= 93/4$$

6. Find an expression for the exact area under $f(x) = x^2 - x$ over the interval $[2, 4]$ and then use a CAS to evaluate it.

$\Delta x = \frac{2}{n}$ $\quad x_i = 2 + \frac{2i}{n}$ $\quad f(x_i) = \left(2 + \frac{2i}{n}\right)^2 + \left(2 + \frac{2i}{n}\right)$

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(2 + \frac{2i}{n}\right)^2 + \left(2 + \frac{2i}{n}\right)$$

$$= 74/3$$

7. Find an expression for the exact area under $f(x) = e^{-x}$ over the interval $[-1, 2]$ and then use a CAS to evaluate it.

$\Delta x = \frac{3}{n}$ $\quad x_i = -1 + 3i/n$

$$A = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} e^{-(-1 + 3i/n)}$$

$$= e - 1/e^2$$

8. Find an expression for the exact area under $f(x) = ln(x + 1)$ over the interval $[1, 4]$ and then use a CAS to evaluate it.

$\Delta x = \frac{3}{n}$ $\quad x_i = 1 + 3i/n$

$$A = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \ln\left(1 + \frac{3i}{n} + 1\right)$$

$$\approx 3.66$$