MTH 251 In Class Worksheet 4

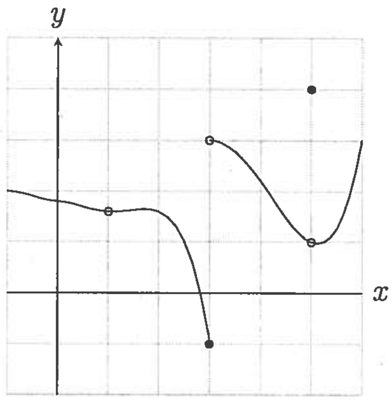
Solutions

Definition: A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Note that this requires both $f(a)$ to be defined and for the limit to exist. No undefined or does not exist weirdness.

1. The figure below shows the graph of a function f. At which numbers is f discontinuous? Use the definition of continuity to justify each discontinuity.



f is discontinuous at 1, 3, & 5

since:

$\lim_{x \to 1} f(x) \approx 1.7$ but $f(1)$ is undefined

$\lim_{x \to 3} f(x)$ d.n.e. but $f(3) = -1$

& $\lim_{x \to 5} f(x) = 1$ while $f(5) = 4$

2. Where are each of the following functions discontinuous? Justify using the definition of continuity.

a. $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x-2)(x+1)}{x-2}$

$= x + 1$ provided $x \neq 2$

f is discontinuos at 2

since $\lim_{x \to 2} f(x) = 3$ but $f(2)$ is undefined.

b. $f(x) = \begin{cases} \dfrac{1}{x^2} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$

$\lim_{x \to 0} f(x) = \infty$ (D.N.E.)

while $f(0) = 1$

so discontinuous at 0.

c. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

f is discontinuous at 2 since $\lim_{x \to 2} f(x) = 3 \neq 1 = f(2)$

d. $f(x) = \lfloor x \rfloor$

Let $k \in \mathbb{Z}$.

Then $\lim_{x \to k^-} f(x) = k - 1$

& $\lim_{x \to k^+} f(x) = k$

so $\lim_{x \to k} f(x)$ DNE

& f is discontinuous at every $k \in \mathbb{Z}$.

Definition: a) A function f is **continuous from the right at a number** a if $\lim_{x \to a^+} f(x) = f(a)$.

b) A function f is **continuous from the left at a number** a if $\lim_{x \to a^-} f(x) = f(a)$.

Definition: A function f is **continuous on an interval** if it is continuous at every number in the interval. If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left* as appropriate.

Idiosyncrasy: Continuous on an interval, continuous at a point, and continuous on a domain are very different things!

3. Let $f(x) = 1 - \sqrt{1 - x^2}$

a. Show that f is continuous on $[-1, 1]$.

Given $a \in [-1, 1]$

$\lim_{x \to a} (1 - \sqrt{1 - x^2})$

$= 1 - \lim_{x \to a} \sqrt{1 - x^2}$

since $a \in$ Dom of the root.

$= 1 - \sqrt{\lim_{x \to a} (1 - x^2)}$

$= 1 - \sqrt{1 - a^2}$

$= f(a)$

b. If f continuous on $\mathbb{R}$?

No. f is undefined outside of $[-1, 1]$.

c. Is f continuous on its domain?

Yes. There are no jumps within the domain.

Theorem: a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

4. Find $\lim\limits_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \quad \text{by continuity}$$

$$= -\frac{1}{11}$$

Theorem: If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} f(g(x)) = f(b)$. In other words, $\lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right)$.

Theorem: If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

5. Where are the following functions continuous?

a. $h(x) = \sin(x^2)$

Sine is continuous everywhere & $\lim\limits_{x \to a} x^2 = a^2$

so $\lim\limits_{x \to a} h(x) = \lim\limits_{x \to a} \sin(x^2)$

$= \sin(a^2)$

So h is continuous everywhere.

b. $F(x) = \ln(1 + \cos x)$

$\ln$ is defined on $(0, \infty)$

& $\lim\limits_{x \to a} (1 + \cos x) = 1 + \cos(a)$

However $1 + \cos(a) = 0$ whenever

$a = \pi + 2\pi k$ where $k \in \mathbb{Z}$

(Note $1 + \cos(a)$ is never less than zero)

Thus $\lim\limits_{x \to \pi + 2\pi k} F(x) = -\infty$

So F is discontinuous whenever

$x = \pi + 2\pi k$, $k \in \mathbb{Z}$

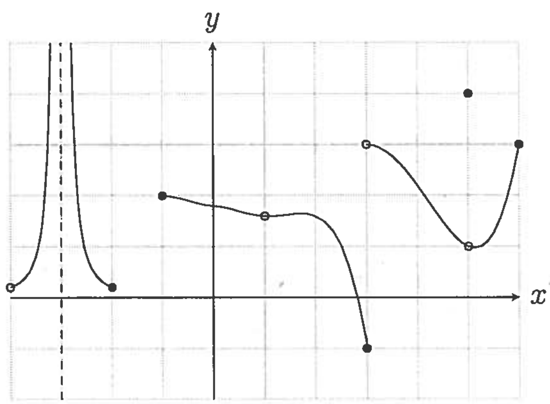
Definition: We say that f has a **removable discontinuity** at a if f is discontinuous at a but $\lim_{x\to a} f(x)$ exists.

Definition: A **hole** occurs when $\lim_{x\to a} f(x)$ exists but $f(a)$ is undefined.

Note: A hole is a particular kind of removable discontinuity.

Definition: I say (Noah says) that f has a **jump discontinuity** at a if $\lim_{x\to a^-}$ exists and $\lim_{x\to a^+}$ exists, but they are not equal to each other.

6. Use the following graph of a function f to answer the following questions.



a. What is the domain of f?

$D = (-4,-3) \cup (-3,-2] \cup [-1,1) \cup (1,6]$

b. At what x values is f discontinuous on $\mathbb{R}$?

when $x \in (-\infty,-4] \cup [-3,-3] \cup [-2,-1] \cup [1,1] \cup [3,3] \cup [5,5] \cup [6,\infty)$

c. At what x values is f discontinuous on the domain of f?

when $x \in \{-2,-1,3,5\}$

d. For each x value in $\mathbb{R}$ for which f is discontinuous explain how it breaks the definition of continuity. Use the words "jump," "removable," and "hole" as appropriate.

For $x \in (-\infty,-4]$ f is undefined

for $x=-3$ f is und. + lim DNE

for $x=-2$ or $x=-1$ one sided limit only

for $x \in (-2,-1)$ f is undefined

for $x=1$ f is undefined (hole)

for $x=3$ jump disc.

for $x=5$ removable (not hole)

7. Draw onto the given coordinate plane a function that satisfies all of the following properties.

i. The only discontinuities on f occur at -2, 0, and 5.

ii. f has two x-intercepts. One at $x = -1$ and one at $x = 3$.

iii. $f(5) = 2$

iv. $\lim_{x \to -\infty} f(x) = 0$

v. $\lim_{x \to -2^-} f(x) = \infty$

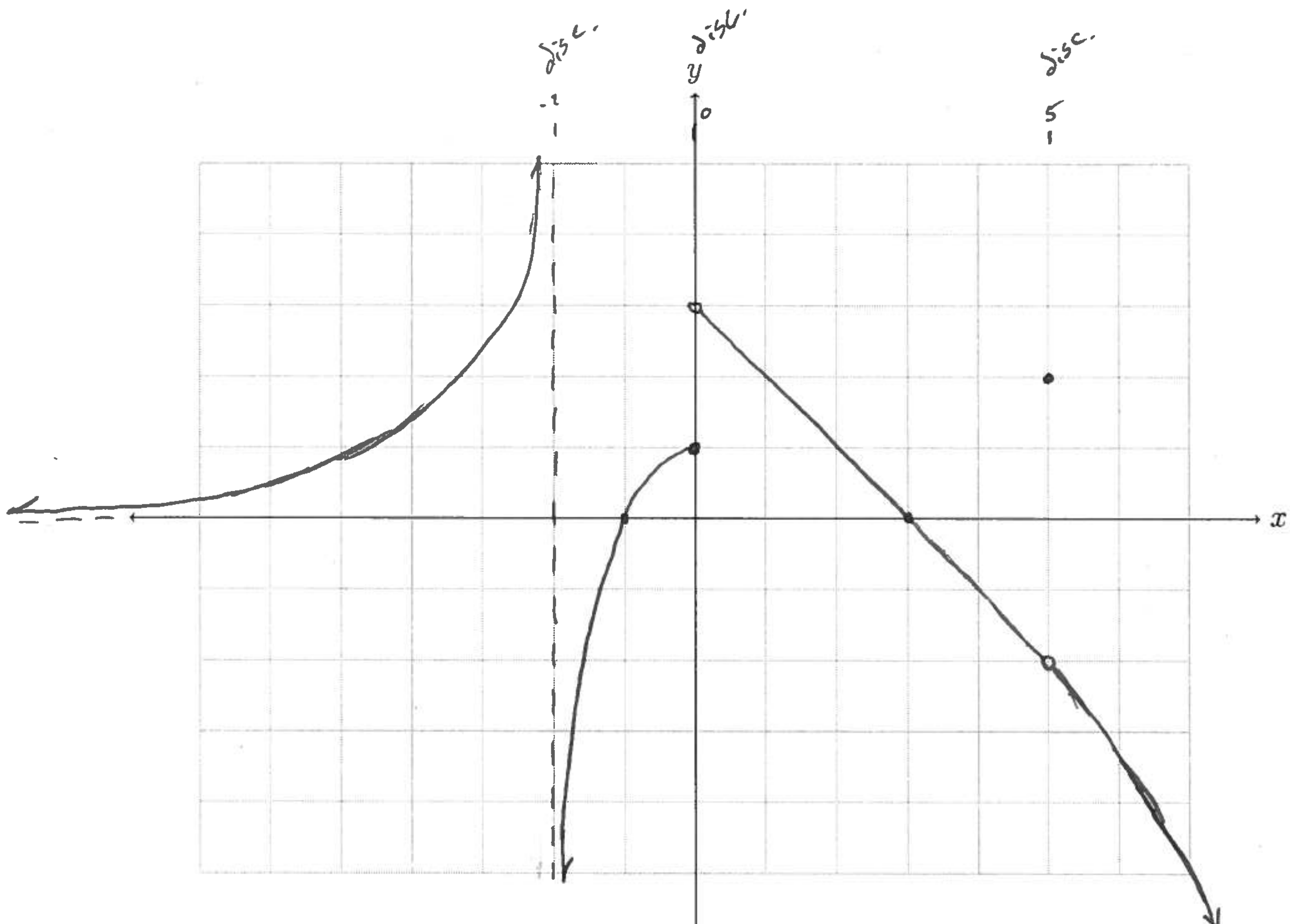
vi. $\lim_{x \to -2^+} f(x) = -\infty$

vii. $\lim_{x \to \infty} f(x) = -\infty$

viii. f has one jump discontinuity.

ix. f has a constant slope of -1 on the interval $(0, 5)$.

x. f has one removable discontinuity.



The Intermediate Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

8. Reference the IVT to show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between $x = 1$ and $x = 2$.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$ which is continuous on $[a,b]$

Note $f(1) = -1$ & $f(2) = 12$

Then 0 is between $f(1)$ & $f(2)$ (& $f(1) \neq f(2)$)

Then, by I.V.T. there exists a number c between 1 & 2 such that $f(c) = 0$.