Math 95 Quadratic Functions Name: ________________________________

§8.1 – 8.6

It's now time to review how to graph quadratic functions. A quadratic function is of the form

$$f(x) = ax^2 + bx + c$$

Remember that a quadratic function graphs a parabola which may be concave up or concave down, has a vertex, a y-intercept, an axis of symmetry, and up to 2 x-intercepts. Let's find these points for $f(x) = -\frac{1}{2}x^2 + x + \frac{3}{2}$ and then use them to graph the parabola making sure to label each point respectively.

Does this parabola open up or down? How do we know?

What is the y-intercept? How did we find it?

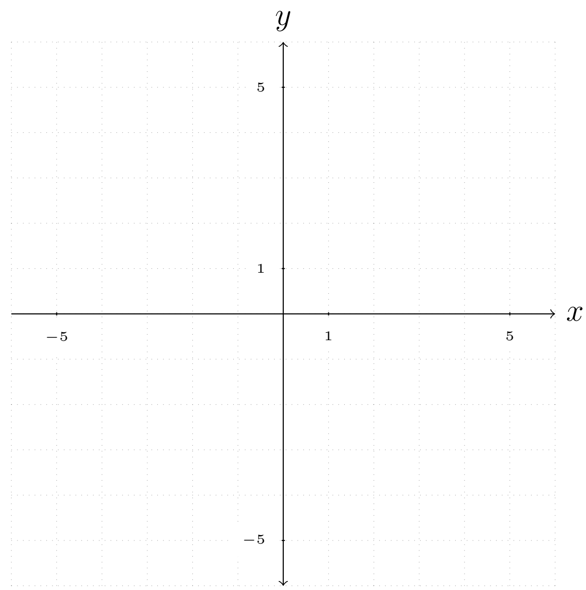
Find the x-intercepts if there are any. What was our method here?

What is the axis of symmetry and the vertex? What is the technique we are using here?

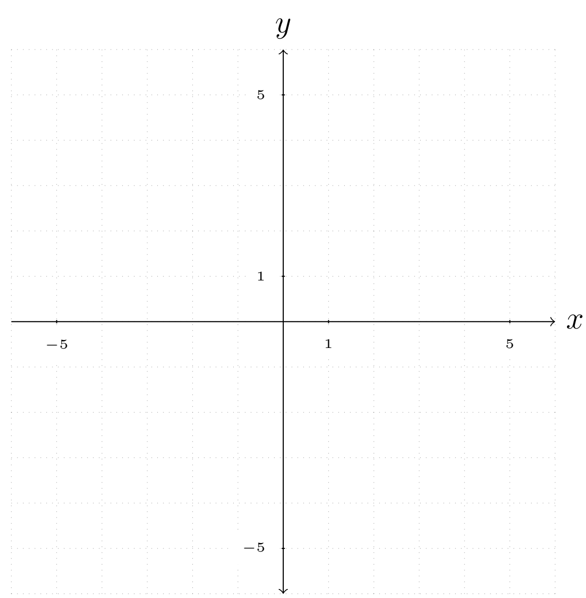
Use the above information to graph $f(x)$ and label the points we've found above.

Here is a couple for you to practice.

1. $f(x) = x^2 - 2x - 3$



2. $f(x) = 3x^2 + 8x - 2$



3. A rectangular pen being constructed for a pet requires 60 feet of fence.

   a. Draw a picture of this situation and then come up with a function $A(x)$ which gives the area of the fence if one side of the pen has length x. Write $A(x)$ in standard form.

   b. What is the maximum area which may be enclosed with this much fencing? What are the dimensions of the pen which allow for this maximum area?

   c. At what side length x, does the pen have a minimum area and what is this minimum area?

   d. Find all necessary points needed to graph $A(x)$ on a domain which is appropriate to the situation. State this domain.

We are next going to look at transformations of the graph $square(x) = x^2$. Before we talk transformations, let's remember what $square(x)$ looks like. What is the vertex, axis of symmetry, x-intercept/y-intercept?

We'll need at least 2 more points after this to graph $square(x)$, we'll actually take a total of 5. You need to memorize these 5 points on this graph.

x	$square(x)$
-2	4
-1	1
0	0
1	1
2	4

Now go ahead and graph $square(x)$ in the space to the right.

We are now going to see how we may *flip*, *stretch*, *compress*, and translate $square(x)$ to get the parabolas for other quadratic functions.

What does the word *translate* mean?

To start let's look at flipping. If I were to reflect $square(x)$ over the x-axis, how have the outputs of my function changed?

Complete the following table then graph $f(x) = -x^2$ in the space to the right.

x	$f(x) = -x^2$
-2	
-1	
0	
1	
2	

So, a negative outside of x^2 takes each y value and makes them negative, thus resulting in an upside down parabola. This is why a negative a value (when looking at the standard form of a quadratic function) means that the parabola is concave down.

Next let's look at stretches and compressions. Begin by filling out the following two tables and graphing the functions in the space to the right.

x	$f(x) = 2x^2$
-2	
-1	
0	
1	
2	

x	$f(x) = \frac{1}{4}x^2$
-2	
-1	
0	
1	
2	

What have we witnessed here?

So, when we have $f(x) = ax^2$ the a value will simply multiply by each y value from $square(x)$ resulting in a stretch or compression and possibly a flip (if a is negative).

Next we need to look at translations. We'll start with the easy ones. Fill out the following tables and graph the function in the space to the right.

x	$f(x) = x^2 + 3$
-2	
-1	
0	
1	
2	

x	$f(x) = x^2 - 1$
-2	
-1	
0	
1	
2	

So here we see $square(x)$ being shifted up and down since we are simply adding or subtracting to the y values of $square(x)$.

Now the hard part. These tables are going to be a bit more difficult, but work in groups and see if you can figure out how to complete them. And, of course, graph them to the right.

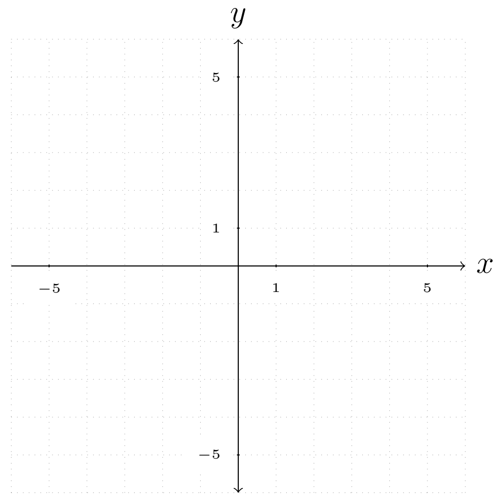
x	$f(x) = (x-2)^2$
	4
	1
	0
	1
	4

x	$f(x) = (x+3)^2$
	4
	1
	0
	1
	4

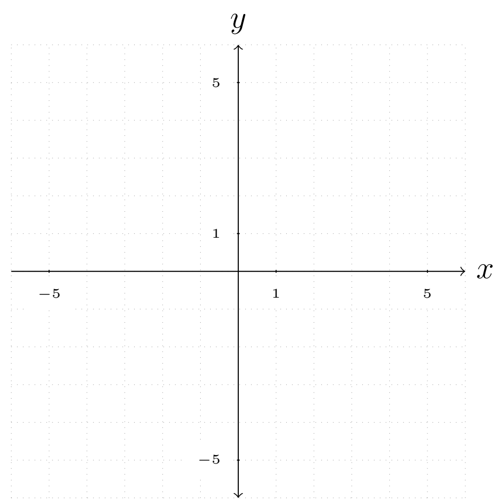
The key to completing these tables is to ask, "what does x need to be in order for y to be ____?" For the second example above what we see is that x needs to be 3 less than it normally would in order to get the same outputs we would in $square(x)$. What this results in is a shift to the left 3 units. The easiest way to remember this is to think that horizontal shifts are *counter*-intuitive.

Let's put these all together with the following examples. For the next two problems identify the transformations and then use them to graph the resultant parabola.

4. $f(x) = (x-1)^2 + 2$



5. $f(x) = 2(x+2)^2 - 3$



These functions that we have been looking at are quadratic functions but they are in *vertex* form rather than standard form.

Vertex Form of a Quadratic Function:

$$f(x) = a(x-h)^2 + k$$

Look back at the last two problems. How do the translations affect the vertex of the parabola?

So when we look at the vertex form of a quadratic function, what we see if that the vertex is (h, k)! This is fantastic! It makes finding the vertex really easy! Use this to quickly determine the vertex for each of the following quadratic functions.

1. $f(x) = (x + 7)^2 - 5$

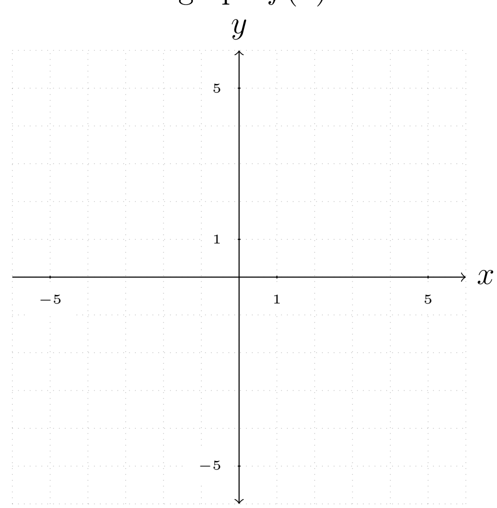
2. $f(x) = \frac{1}{2}(x - 5)^2 + 19$

3. $f(x) = -3(x - \frac{1}{4})^2 - \frac{2}{3}$

4. $f(x) = -(x + \pi)^2 - 2.87$

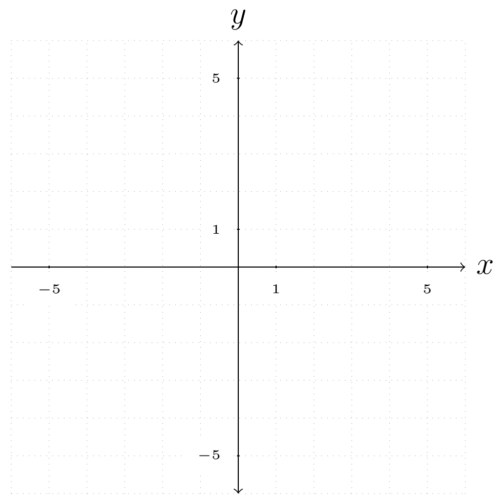
Now, if I want to graph these functions I really just need the x and y intercepts and I'll be home free. Let's do this with $f(x) = \frac{1}{3}(x - 3)^2 + 6$.

First the y-intercept is at $f(0) = -\frac{1}{3}(0 - 3)^2 + 6 = 3$. To get the x intercepts we set $f(x) = 0$ and solve for x. Go ahead and do this and then graph $f(x)$.



Here's another for you to practice:

5. $f(x) = 2(x+1)^2 - \frac{9}{2}$

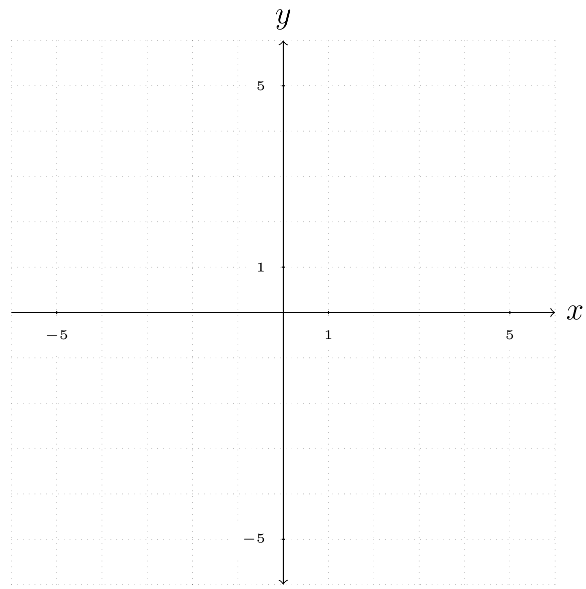


What we are witnessing is the miracle of the vertex form. All of the sudden, factoring and quadratic formula stuff get thrown right out the window. However, there's a problem. What if the function isn't given in vertex form? Well, guess what, we can *convert* standard form to vertex form. Let's look at the process.

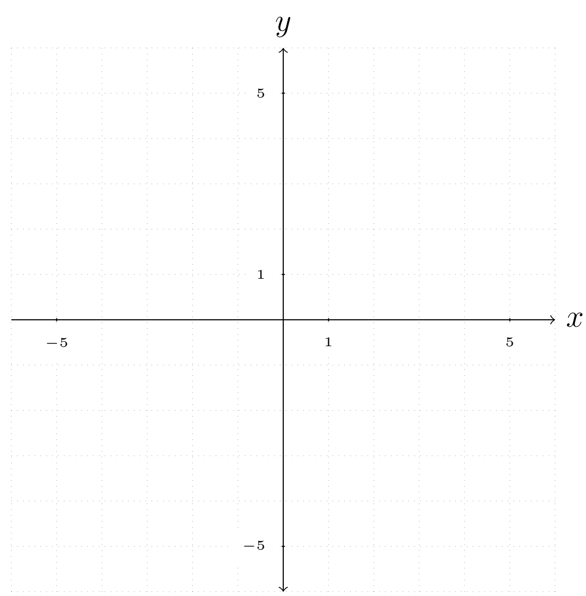
Given $f(x) = \frac{1}{2}x^2 + 3x - 6$, convert $f(x)$ into vertex form and then use this form to graph $f(x)$ by finding the vertex and x and y intercepts.

Here's a couple for you to practice:

6. $f(x) = 2x^2 - 5x + 2$



7. $f(x) = \frac{1}{3}x^2 + 2x + 1$



This technique we've been using is called *completing the square*. We can also utilize this technique in order to solve quadratic equations. In fact, we really just did that in the two examples above. We converted the function into vertex form and then found the x-intercepts. Let's do this a couple more times, just for good measure.

Solve for x in the following quadratic equations.

1. $0 = 3x^2 - 5x - 6$

2. $18 = 2x^2 - 12x$

3. $0 = \frac{3}{4}x^2 + \frac{3}{2}x + 1$

So now that we've seen the process of completing the square in action a few times, let's use the process to do something fantastic: Prove the Quadratic Formula!

Remember, the quadratic formula gives us solutions to any equation of the form $0 = ax^2 + bx + c$. We know use completing the square to find the solutions:

$$\begin{aligned}
0 &= ax^2 + bx + c \\
0 &= a\left(x^2 + \frac{b}{a}x\right) + c \\
0 &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
0 &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
0 &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c \\
0 &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
\frac{b^2}{4a} + c &= a\left(x + \frac{b}{2a}\right)^2 \\
\frac{b^2}{4a^2} + \frac{c}{a} &= \left(x + \frac{b}{2a}\right)^2 \\
\pm\sqrt{\frac{b^2}{4a^2} + \frac{c}{a}} &= x + \frac{b}{2a} \\
\pm\sqrt{\frac{b^2}{4a^2} + \frac{4ac}{4a^2}} &= x + \frac{b}{2a} \\
\pm\sqrt{\frac{b^2 + 4ac}{4a^2}} &= x + \frac{b}{2a} \\
\pm\frac{\sqrt{b^2 + 4ac}}{2a} &= x + \frac{b}{2a} \\
-\frac{b}{2a} \pm \frac{\sqrt{b^2 + 4ac}}{2a} &= x \\
x &= \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}
\end{aligned}$$

Woo Hoo! There it is! Alright, enough celebration, let's introduce imaginary numbers. Wait, what? Yes, that's right, *imaginary* numbers...

Negative numbers seem second nature to us in our society. We deal with negative numbers when referring to being below sea level, under par in golf and when we have debt. However, negative numbers were not always so easily accepted. In parts of China, negative numbers were first recorded as being accepted by mathematicians in the 2nd century AD. European mathematicians didn't widely accept negative numbers until the $17th$ century AD. That's right, less than 400 years ago, people didn't believe in negative numbers. What this means is that if a mathematician was presented with the equation

$$4x + 20 = 0$$

they would say that it had *no solutions*. The concept of a negative number was introduced to deal with equations such as the one presented above. This was a HUGE leap of faith for someone to *believe* these numbers existed during that time period. Today, it's like second nature.

But, what does this have to do with imaginary numbers? Well, for another hundred years or so after negative numbers became widely accepted, equations such as

$$x^2 + 5 = 0$$

were still believed to have *no solutions*, much as $4x + 20 = 0$ was considered to have no solutions a hundred years prior. However, the concept of a solution to $x^2 + 5 = 0$ came about in the 18th century when mathematicians began to familiarize themselves with what we now call *imaginary numbers*. Much like negative numbers were created to solve $4x + 20 = 0$, imaginary numbers were created to solve $x^2 + 5 = 0$.

If we solve for x in the above equation we get $x = \pm\sqrt{-5}$. Now, we know that $\sqrt{-5}$ is not a real number, that is, it doesn't represent some length in the real world, nor does it represent the negative version of some length in the real world. However, it *is* a number and the kind of number we call it is an *imaginary* number. Really, the word "imaginary" was a poor choice of name for these kinds of numbers for this provides the connotation that they are fake. This is not the case, they are just as much a number as any other number you come across - just not nearly so easy to conceptualize.

Let's now introduce some new notation: i. i is defined to be the *positive* solution to $x^2 = -1$ or, in other terms $i = \sqrt{-1}$. In practice we would then use this notation to rewrite $\sqrt{-5} = \sqrt{5}i$, $3\sqrt{-4} = 6i$ and $2\sqrt{-12} = 4\sqrt{3}i$.

Try rewriting the following numbers using the imaginary unit i:

4. $\sqrt{-7}$

5. $\sqrt{-4}$

6. $3\sqrt{-9}$

7. $5\sqrt{-18}$

The next thing to introduce is the *complex number*. This is simply an imaginary number added to a real number. It is a number which may be written in the form

$$a + bi$$

where a and b are real numbers.

For example, $5 + 2i$, $-3 - 4i$, $\pi - \sqrt{2}i$ and $-2.3 - 5.1i$ are all complex numbers. We call the term without the i the *real part* and the term with the i the *imaginary part*.

Is 5 a complex number?

How about $-\sqrt{7}i$?

We now return to the quadratic formula. Use the quadratic formula to solve the following equations. Write your solutions in set notation and in the form $a + bi$.

8. $2x^2 + x + 3 = 0$

9. $5x^2 - 7x + 2 = 0$

10. Suppose we have the function $f(x) = 2x^2 + x + 3$. How many x-intercepts does this function have? (Use the quadratic formula here)

11. What about $f(x) = 3x^2 - 6x + 3$?

12. And $f(x) = x^2 - 5x + 2$?

13. What is it about each of these functions that tells us how many x-intercepts it has?

The part of the quadratic formula inside the square root, $b^2 - 4ac$ is called the *discriminant*. By calculating the discriminant we can quickly determine how many x-intercepts a quadratic function has. It also will quickly tell us how many real solutions and imaginary solutions we have.

14. How many imaginary and how many real solutions does $x^2 + 3x - 7 = 0$ have?

15. $25x^2 - 20x + 1 = 0$?

16. $3x^2 - 9x + 2 = 0$?

A quadratic inequality may be written in the form

$$ax^2 + bx + c \geq 0, \qquad ax^2 + bx + c \leq 0, \qquad ax^2 + bx + c > 0 \text{ or } ax^2 + bx + c < 0.$$

In order to solve an inequality we should *think* about what's going on in an individual situation. Let's take a look at a few.

Suppose we want to solve $x^2 + 3x - 4 > 0$. We look at $f(x) = x^2 + 3x - 4$ and think about it's behavior. Is the corresponding parabola concave up or concave down? How many x-intercepts does it have? Do we want to talk about the part above or below the x-axis?

Now, what are the x-intercepts? How can we use these to determine the set of solutions to our inequality? Finally, what is our set of solutions?

Try this thinking to try and solve the following inequalities.

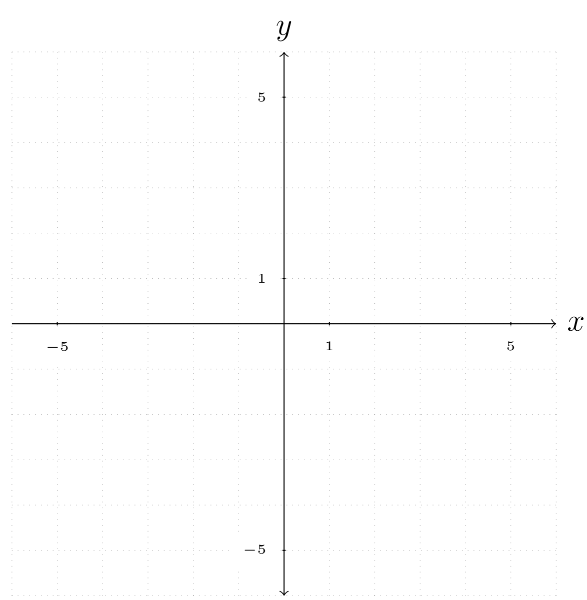
17. $x^2 - 5x + 7 > 0$

18. $3x^2 + 4x - 5 < 0$

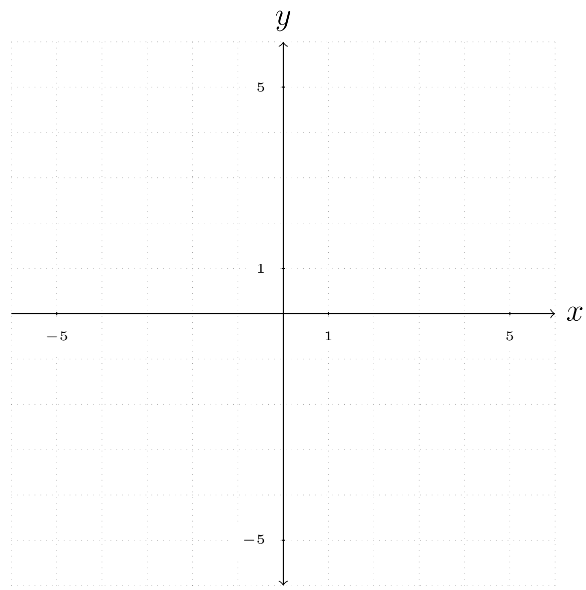
19. $-2x^2 + 3x - 4 > 0$

Lets see how using graphing to solve these inequalities allows us to actually see these solutions.

When solving $x^2 + 3x - 4 > 0$, let $y_1(x) = x^2 + 3x - 4$ and $y_2(x) = 0$. We saw above that $y_1(x)$ has x-intercepts of $(4, 0)$ and $(-1, 0)$, we can easily see that the y-intercept is $(0, -4)$ and so to complete the graph we just need the vertex. Go ahead and find the vertex using your favorite method and then graph $y_1(x)$ and $y_2(x)$ in the coordinate plane below.



20. Solve $-2x^2 + 3x - 4 > 0$ by graphing.



21. A projectile is launched into the air from 50 feet up and with an initial vertical velocity of 200 feet per second. The height of the projectile may be modeled by the function

$$h(t) = -16t^2 + 200t + 50.$$

During what time period is the projectile have an elevation grater than 500 feet?