Math 111 WS 6 Composite Functions

Name: Solutions

1. Suppose $f(x) = 3 - 2x^2$ and $g(x) = 9 - 6x$. Find $(f+g)(2)$, $(f-g)(2)$, $(f \cdot g)(-3)$, $\left(\frac{f}{g}\right)(-1)$ and $\left(\frac{g}{f}\right)(1)$.

$$(f+g)(2) = f(2) + g(2)$$
$$= 3 - 2(2)^2 + 9 - 6(2)$$
$$= 3 - 8 + 9 - 12 = -8$$

$$(f-g)(2) = f(2) - g(2)$$
$$= -5 - (-3) = -2$$

$$(f \cdot g)(-3) = f(-3) \cdot g(-3)$$
$$= -15 \cdot 27 = -405$$

$$\left(\frac{f}{g}\right)(-1) = \frac{f(-1)}{g(-1)}$$
$$= \frac{1}{15}$$

$$\left(\frac{g}{f}\right)(1) = \frac{g(1)}{f(1)}$$
$$= \frac{3}{1} = 3$$

2. Suppose $f(x) = 2x^2 + 3$ and $g(x) = \sqrt{-3x+1}$. Find $(f+g)(x)$ and $\left(\frac{f}{g}\right)(x)$. State the domain of each.

$$(f+g)(x) = f(x) + g(x)$$
$$= 2x^2 + 3 + \sqrt{-3x+1}$$

$$\left(\frac{f}{g}\right)(x) = \frac{2x^2+3}{\sqrt{-3x+1}}$$
$$= \frac{(2x^2+3)\sqrt{-3x+1}}{-3x+1}$$

3. Suppose $f(x) = \dfrac{1}{\sqrt{2x+1}}$ and $g(x) = \dfrac{1}{5x^2+1}$. Find $(f-g)(x)$ and $\left(\frac{g}{f}\right)(x)$. State the domain of each.

$$(f-g)(x) = f(x) - g(x)$$
$$= \frac{1}{\sqrt{2x+1}} - \frac{1}{5x^2+1}$$
$$= \frac{5x^2+1-\sqrt{2x+1}}{(5x^2+1)\sqrt{2x+1}}$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$$
$$= \frac{1/5x^2+1}{1/\sqrt{2x+1}}$$
$$= \frac{\sqrt{2x+1}}{5x^2+1}$$

4. Suppose $f(x) = 2x^2 - x + 3$. Find $f(2)$, $f(2+h)$, $f(2+h) - f(2)$, $f(a)$, $f(a+1)$, $f(x+h)$, and $f(x+h) - f(x)$.

$f(2) = 2(2)^2 - 2 + 3 = 9$

$f(a) = 2a^2 - a + 3$

$f(a+1) = 2(a+1)^2 - (a+1) + 3$
$= 2(a^2 + 2a + 1) - a - 1 + 3$
$= 2a^2 + 3a + 4$

$f(x+h) = 2(x+h)^2 - (x+h) + 3$
$= 2(x^2 + 2xh + h^2) - x - h + 3$
$= 2x^2 + 4xh + 2h^2 - x - h + 3$

$f(2+h) = 2(2+h)^2 - (2+h) + 3$
$= 2(4 + 4h + h^2) - 2 - h + 3$
$= 2h^2 + 7h + 9$

$f(2+h) - f(2) = (2h^2 + 7h + 9) - 9$
$= 2h^2 + 7h$

$f(x+h) - f(x)$
$= \cancel{2x^2} + 4xh + 2h^2 - \cancel{x} - h + \cancel{3} - (\cancel{2x^2} - \cancel{x}$
$= 4xh + 2h^2 - h$

5. Given the following functions, find $\dfrac{f(1+h) - f(1)}{h}$ and $\dfrac{f(x+h) - f(x)}{h}$.

a. $f(x) = 3x - 4$

$$\frac{f(1+h) - f(1)}{h} = \frac{3(1+h) - 4 - (-1)}{h} = \frac{3h}{h} = 3$$

$$\frac{f(x+h) - f(x)}{h} = \frac{3x + 3h - 4 - (3x - 4)}{h} = \frac{3h}{h} = 3$$

b. $f(x) = 3x^2 - 2x + 1$

$$\frac{f(1+h) - f(1)}{h} = \frac{3(1+h)^2 - 2(1+h) + 1 - (2)}{h}$$
$$= \frac{3(h^2 + 2h + 1) - 2 - 2h - 1}{h}$$
$$= \frac{3h^2 + 4h}{h} = 3h + 4$$

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 - 2(x+h) + 1 - (3x^2 - 2x +}{h}$$
$$= \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{2x} - 2h + \cancel{1} - \cancel{3x^2} + \cancel{2x} - \cancel{1}}{h}$$
$$= \frac{6xh + 3h^2 - 2h}{h} = 6x + 3h - 2$$

6. Suppose that $f(x) = 2x^2 - 3$ and $g(x) = 4x$. Find $(f \circ g)(1)$, $(g \circ f)(1)$, $(f \circ f)(-2)$, and $(g \circ g)(-1)$.

$(f \circ g)(1) = f(g(1))$
$= f(4) = 29$

$(g \circ f)(1) = g(f(1))$
$= g(-1) = -4$

$(f \circ f)(-2) = f(f(-2))$
$= f(5) = 47$

$(g \circ g)(-2) = g(g(-2))$
$= g(-8) = -32$

7. Suppose that $f(x) = x^2 + 3x - 1$ and $g(x) = 2x + 3$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x) = f(g(x))$
$= f(2x+3)$
$= (2x+3)^2 + 3(2x+3) - 1$
$= 4x^2 + 12x + 9 + 6x + 9 - 1$
$= 4x^2 + 18x + 17$

$(g \circ f)(x) = g(f(x))$
$= g(x^2 + 3x - 1)$
$= 2(x^2 + 3x - 1) + 3$
$= 2x^2 + 6x + 1$

8. Suppose that $f(x) = \dfrac{2}{x-2}$ and $g(x) = \dfrac{4}{2x-5}$. Find $(f \circ g)(x)$ and $(f \circ f)(x)$ and state the domain of each.

$(f \circ g)(x) = f(g(x))$
$= f\left(\dfrac{4}{2x-5}\right)$
$= \dfrac{2}{\frac{4}{2x-5} - 2} \cdot \dfrac{2x-5}{2x-5}$
$= \dfrac{4x-10}{4 - 4x + 10} = \dfrac{4x-10}{-4x+14}$
$= \dfrac{2x-5}{-2x+7}$

$D = \{x \mid x \neq \frac{5}{2}, \frac{7}{2}\}$

$(f \circ f)(x) = f(f(x))$
$= f\left(\dfrac{2}{x-2}\right)$
$= \dfrac{2}{\frac{2}{x-2} - 2} \cdot \dfrac{x-2}{x-2}$
$= \dfrac{2x-4}{2 - 2x + 4} = \dfrac{2x-4}{-2x+6}$
$= \dfrac{x-2}{-x+3}$

$D = \{x \mid x \neq 3, 2\}$

9. Suppose that $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$. Find $f \circ g$ and $g \circ g$ and state the domain of each.

$(f \circ g)(x) = f(g(x))$
$= f\left(\dfrac{4}{x-1}\right)$
$= \dfrac{1}{\frac{4}{x-1} + 2} \cdot \dfrac{x-1}{x-1}$
$= \dfrac{x-1}{4 + 2x - 2}$
$= \dfrac{x-1}{2x+2}$

$D = \{x \mid x \neq 1, -1\}$

$(g \circ g)(x) = g\left(\dfrac{4}{x-1}\right)$
$= \dfrac{4}{\frac{4}{x-1} - 1} \cdot \dfrac{x-1}{x-1}$
$= \dfrac{4x-4}{4 - x + 1}$
$= \dfrac{4x-4}{-x+5}$

$D = \{x \mid x \neq 1, 5\}$

10. Find functions f and g such that $f \circ g = H$ if $H(x) = (x^2+1)^{50}$. In fact, find multiple solutions to this exercise.

Want $f(g(x)) = (x^2+1)^{50}$

Sol 1: $g(x) = x^2+1$, $f(x) = x^{50}$

Sol 2: $g(x) = x$, $f(x) = (x^2+1)^{50}$

Sol 3: $g(x) = x^2$, $f(x) = (x+1)^{50}$

11. Find functions f and g such that $f \circ g = H$ if $H(x) = \dfrac{1}{x+1}$. Again, find multiple solutions to this exercise.

Want $f(g(x)) = \dfrac{1}{x+1}$

Sol 1: $g(x) = x$, $f(x) = \dfrac{1}{x+1}$

Sol 2: $g(x) = x+1$, $f(x) = \dfrac{1}{x}$

Sol 3: $g(x) = \dfrac{1}{x+1}$, $f(x) = x$

12. The price p, in dollars, of a certain commodity and the quantity x sold obey the demand equation

$$p = -\frac{1}{5}x + 200 \qquad 0 \le x \le 1000$$

Suppose that the cost $C = \dfrac{\sqrt{x}}{10} + 400$

Assuming that all items produced are sold, find the cost C as a function of the price p.

$p = -\frac{1}{5}x + 200$

$\Rightarrow \quad -\frac{1}{5}x = p - 200$

$\Rightarrow \quad x = 1000 - 5p$

$C(x) = \dfrac{\sqrt{x}}{10} + 400$

$C(p) = \dfrac{\sqrt{1000-5p}}{10} + 400$